Competition and Errors in Breaking News

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Abstract

Reporting errors are endemic to breaking news, even though accuracy is prized by consumers. I present a continuous-time model to understand the strategic forces behind such reporting errors. News firms are rewarded for reporting before their competitors, but also for making reports that are credible in the eyes of consumers. Errors occur when firms *fake*, reporting a story despite lacking evidence. I establish existence and uniqueness of an equilibrium, which is characterized by a system of ordinary differential equations. Errors are driven by both a lack of commitment and by competition. A lack of commitment power gives rise to errors even in the absence of competition: firms are tempted to fake after their credibility has been established, capitalizing on the inability of consumers to detect fake reports. Competition exacerbates faking by engendering a preemptive motive. In addition, competition introduces observational learning, which causes errors to propagate through the market. The equilibrium features rich dynamics. Firms become gradually more credible over time whenever there is a preemptive motive. The increase in credibility rewards firms for taking their time, and thus endogenously mitigates the haste-inducing effects of preemption. A firm's behavior will also change in response to a rival report. This can take the form of a *copycat effect*, in which one firm's report triggers an immediate surge in faking by others.

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1. Introduction

What a newspaper needs in its news, in its headlines, and on its editorial page is terseness, humor, descriptive power, satire, originality, good literary style, clever condensation, and accuracy, accuracy, accuracy!

— Joseph Pulitzer

Accuracy is often considered to be the core tenet of news media. This belief is widely held by consumers of news: when asked in a 2018 Pew survey, the majority of respondents listed accuracy as a primary function of news, valuing it over thorough coverage, unbiasedness, and relevance.

Despite this, public perceptions of news accuracy are not favorable. In a 2020 Survey, 38% of respondents stated that they go into a news story thinking it will be largely inaccurate. While many factors may contribute to this skepticism, consumers express particular concern about hasty reporting: 53% of respondents believe that news breaking too quickly is a major source of errors.

These concerns are supported by a multitude of instances in which news media have made major factual errors. In the immediate aftermath of the 9/11 attacks, cable news stations made multiple statements that were false: NBC reported an explosion outside the pentagon, CNN reported a fire outside the national mall, and CBS claimed the existence of a car bomb outside the state department. Erroneous reporting has been endemic to terrorist attacks in general, with news media misidentifying perpetrators or other key details of the Boston bombings, Sandy Hook massacre, London bombings, and Oklahoma City bombings. Furthermore, such errors are not limited to terrorist attacks. Notoriously, in 2004 CBS news, under the direction of Dan Rather, published the Killian Documents, a collection of memos which called into question George W. Bush's military record. These documents could never be authenticated and were widely believed to be forged. More recent media blunders are ever present: in 2017, ABC news falsely reported that Michael Flynn would testify that Donald Trump had directed him "to make contact with the Russians." In 2019, ABC News headlined its nightly news broadcast with what it claimed to be exclusive footage of the ongoing air strikes on Syria. It was later uncovered that this footage was in fact taken at a machine gun convention in Oklahoma.

While such errors are commonplace, they are also costly to news firms. For one, exposure of errors can be reputationally damaging. This was acutely true of the *Rolling Stone* scandal, in which the magazine falsely accused a group of University of Virginia students of sexual assault. Not only was the journalistic failure widely reported by other firms, the error resulted in several publicized lawsuits against the magazine. Furthermore, major errors often lead firms to part with valuable journalists in an effort to protect their reputations. This was evident in the terminations of Dan Rather and Brian Ross —both lead journalists at major news stations—following their respective reporting blunders.

The objective of this paper is twofold. First, I seek to understand why reporting errors are pervasive despite their costliness to firms. In particular, I explore how *strategic* forces can induce firms to commit errors that are completely avoidable. My second objective is to understand *when* reporting errors are most probable, and relatedly, when firms are less trustworthy. That is, I seek to understand both the dynamics of reporting errors and the environmental factors that can make them more prevalent.

Model To answer these questions, I present a dynamic model of breaking news. I consider a continuous-time setting where multiple firms dynamically and privately learn about a story and must choose if and when to report it. Firms learn by seeking confirmation that the story is true. Reporting errors occur when firms *fake*, i.e., report the story despite lacking confirmation. Because reports are publical, firms also learn by observing the reports of their competitors. I thus account for an important feature of the newsroom setting: firms learn privately but also observationally.

Firms in this model seek viewership. Error-prone reporting conflicts with this objective, and is thus costly to the firm, in two ways. First, errors harm firms ex post (after they have been exposed). This ex-post cost captures the detrimental effect of errors on a firm's future livelihood. Importantly, error-prone reporting is also costly ex ante (before errors can be unearthed). This is due to the fact that a firm's viewership hinges on its *credibility*, i.e., the consumer's belief that the report is not fake. This belief is formed rationally with knowledge of the firm's reporting strategy: firms who fake more achieve lower credibility in equilibrium. By making this assumption, I take the stance that a story is valued to the extent that there is trust in a firm's journalistic standards, a notion that is informed by consumers' demonstrated preference for accurate news.

Finally, this model accounts for one of the most salient qualities of the breaking news problem: competition. All else equal, a firm who preempts its rivals (e.g., by being the first to report) is rewarded with greater viewership. This allows us to understand the impact of competition on the propensity of firms to err. Doing so is especially pertinent given the rise of digital news. Since the ascent of the internet, there has been a documented shift from print to digital news.[1] This shift has arguably contributed to a news industry where firms feel greater pressure to get stories out quickly in order to beat out competitors. This is due to the fact that, while print news is limited to daily publication at most, digital news faces no such constraints.[2] By considering a continuous-time setting, one can better understand 24-hour news environment, where preemptive concerns are not only present but ceaseless.

Analysis I analyze this model, establishing both existence and uniqueness of an equilibrium. Under this equilibrium, fake reports do not occur at set times, but are rather distributed continuously over time. This mixing implies an *indifference condition*: at any time in which the firm must fake, it must be indifferent between faking immediately and after some short wait. Formally, this condition implies an ordinary differential equation (ODE) on the firm's reporting behavior. I thus show that the equilibrium is characterized by a system of ODEs, a result which is central to our analysis and guides many of the economic implications that follow.

Economic Implications I find that errors are strategic responses to two features of the news environment: a lack of commitment by firms, and competition.

To this end, I begin by showing that competition alone is not responsible for reporting errors. In particular, if the ex-post cost of error is relatively small —because consumers are less aware or critical of them —even a monopolist will fake. Such errors are driven by a firm's *inability to commit* to a reporting strategy: a firm is tempted fake after its credibility has been assessed. This is due thte fact that firms cannot observe whether a firm is faking, and thus the firm is not directly punished for doing so. I substaniate the notion that a lack of commitment causes errors by proving that a firm who can commit will always report truthfully, and thus never err.

I then show that competition exacerbates errors, and does so through two separate channels. First, competition can give rise to a preemptive motive in equilibrium: firms have an incentive to speed up their reporting in order to beat out competitors. This incentive for speed induces firms to fake and thus err. Second, competition causes errors through another, less obvious channel: observational learning. When one firm reports a story, other firms become more confident that the story is true. This increased confidence in turn yields firms more likely to fake. I thus find that observational learning exacerbates errors not by

[1] While 16% of 2018 survey respondents often receive news from print newspapers, 33% do so from news websites.

[2] This is also true of TV news, which remains the most popular news medium in the United States.

giving rise to them in the first place, but by causing existing errors to propagate through the market.

This paper also sheds light on the dynamics of reporting behavior and credibility. These dynamics take two different forms in equilibrium: gradual changes that happen in the absence of new reports and discrete changes that occur in response to a new report.

I first show that firms become gradually more truthful —i.e., less inclined to fake—as time passes. Furthermore, firms become more credible over time whenever preemptive concerns are present. In other words, consumers are less trusting of reports that are made quickly. This model thus justifies consumers' expressed concerns about hasty reporting. The reason for this gradual improvement in credibility lies in the firms incentives. The risk of being preempted introduces an endogenous cost to delay. That is, the firm must somehow be compensated for this cost to ensure that its indifference condition is satisfied. This is achieved by means of increasing credibility. That is, increasing credibility mitigates the haste-inducing effects of preemption.

In addition to this gradual increase in credibility, dynamics can take a second form: discrete changes in a firm's reporting behavior and credibility in response to a rival report. This can entail a *copycat effect*, in which one firm's report causes an instantaneous boost in faking by others. The copycat effect implies that when one firm's report is quickly repeated by other firms, such follow-up reports will often lack credibility because they are not independently verified. It illustrates that firms can herd on both the reports themselves and the *timing* of their reports. This provides an explanation for the "clustering" of reporting errors that can occur in breaking news.[3]

In addition to these core results, I consider comparative statics and an extension of the model. I find that, unsurprisingly,credibility is improved by both a higher ex-post cost of error and a higher learning ability. I also further explore the role of competition by considering the marginal effect of an additional firm in the market. Whenever preemptive concerns are present, adding a competitor will make each individual firm more likely to fake early on by increasing the preemptive threat they face. However, this is mitigated later on by the effects of observational learning: existing firms are able to learn that the story is false more quickly by observing the silence of an additional competitor, which will yield them less willing to fake. Finally, I extend the model to allow for heterogeneity in firms' ability to learn. This extended model gives rise to an intuitive result: firms with greater ability to learn are also more credible in equilibrium. Though there are many potential

[3] Examples of this include the reporting surrounding the Boston bombings and the 2000 US presidential election.

reasons why ability and accuracy can correlate in the market for news, this model provides a novel explanation: firms with lower ability face a greater preemptive threat, and are thus more willing to fake.

Related Literature. The preemption literature has modeled a variety of scenarios, including R&D races (Fudenberg, Gilbert, Stiglitz, and Tirole (1983)), technology adoption (Fudenberg and Tirole (1985)), the strategic exercise of options (Grenadier (1996)), and financial bubbles (Abreu and Brunnermeier (2003)). This paper contributes to this literature in two key ways. The first is in the endogeneity of the payoff function. In the existing literature, a player's decision to preempt does not affect its underlying payoff function. That is, the benefit of preempting may be stochastic (e.g., Grenadier (1996)), but it is exogenous. In my setting, however, a firm's payoff from reporting hinges on the consumer's beliefs about its reporting behavior. Such beliefs are important in the market for news because consumers may not be able to immediately observe the quality of a news report, e.g., whether it was verified before being reported. This assumption has implications for the nature of the firm's incentives. While in the existing literature, players earn some exogenous benefit from delaying their actions which counteracts the incentive to preempt, this is not true in our setting. Rather, I find that even if no such benefit exists exogenously, it will arise endogenously.

This paper is not the first to consider observational learning in a preemption setting. In Hopenhayn and Squintani (2011), firms can only observe their own payoffs, and thus draw inferences about the payoffs of their competitors by observing when and whether they act. Meanwhile, in Bobtcheff, Bolte, and Mariotti (2017), players receive breakthroughs which are privately observed, and thus at every moment are uncertain about how much competition they face. In contrast, I assume that firms learn observationally about their *own* payoffs, namely whether publishing a story will result in error. It is for this reason that observational learning causes firms to herd on not only on the decisions of their opponents but also on the timing of these decisions. In this sense, this paper also connects to the literature on herding with endogenously-timed decisions (Gul and Lundholm (1995), Chamley and Gale (1994), Levin and Peck (2008)). In particular, the notion that an action by one individual can trigger others to quickly follow suit arises in Gul and Lundholm (1995). While such behavior is efficient in their setting, that is not the case in ours, where it can cause errors to propagate through the market.

To my knowledge, there are two other papers that study preemption in breaking news: Lin (2014) and Pant and Trombetta (2019). In both settings, a firm benefits in some way from being the first to report, and in Lin (2014), incurs some cost of error. However, neither of these works account for the two novel features highlighted above, i.e., the role of credibility and observational learning. It is because I account for these additional features that I find errors to be driven by not preemption alone, but also a lack of commitment power and observational learning. Furthermore, this paper differs from these other works by giving rise to dynamics in the firm's reporting behavior, including herding.

This paper also contributes to a broad literature on the impact of competition on news quality. This literature is surveyed by Gentzkow and Shapiro (2008), with more recent contributions by Liang, Mu, and Syrgkanis (2021), Galperti and Trevino (2020), Chen and Suen (2019), and Perego and Yuksel (2018). Chen and Suen (2019) and Galperti and Trevino (2020) specifically consider the effects of competition on news accuracy. In both papers, firms compete for the attention of consumers and face constraints or costs to accuracy. Meanwhile, in my setting, accuracy is not intrinsically costly. Rather, accurate reporting entails an indirect cost, namely that of being preempted. I contribute more generally to this literature in two ways. First, I consider the effects of competition on a different notion of accuracy, namely the prevalence of factual errors. Second, this paper also sheds light on the dynamics of firm behavior. This allows one to understand the effects of competition on not only on news quality as a whole, but also on its time path.

Finally, this paper connects broadly to the literature on the strategic provision of information. Unlike frameworks where a sender seeks to induce a particular action from receivers (Crawford and Sobel (1982), Kamenica and Gentzkow (2011)), firms in my model treat information as a good, aiming to maximize its appeal to consumers. This notion underlies the literature on demand-driven media bias. In Mullainathan and Shleifer (2005), firms bias their reports in an appeal to consumers' preferences for having their beliefs confirmed. Meanwhile, in Gentzkow and Shapiro (2006) bias arises purely in response to reputational concerns, and is thus driven by an aim for long-term profitability. My framework accounts for both the short-term and long-term objectives of a news firm. This sheds light on an intertemporal tradeoff faced by news media: low-quality reporting may benefit a firm in the short run, but can cause damage in the long run. Separately, I note that the kind of deception firms engage in shares common threads with other work. The notion of faking is also studied in Boleslavsky and Taylor (2020) in a competition-free setting that encorporates discounting. Furthermore, the endogenous Poisson arrival of inaccurate information, a feature our equilibrium exhibits, also arises in Che and Hörner (2018), and takes the form of "spamming" by recommender systems.

Outline The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 is dedicated to characterizing its equilibrium, first considering the monopoly benchmark and then encorporating competition. In Section 4, I present the core economic implications of this equilibrium, which pertain to the effects of competition and equilibrium dynamics. In Section 5, I present comparative statics. Section 6 considers an extension of the model in which firms have heterogenous learning abilities. Finally, Section 7 concludes. All formal proofs are relegated to the Appendix.

2. A model of breaking news

There are $N \geq 1$ firms, indexed by i, and one consumer. Time, which is continuous and has an infinite horizon, is denote by $t \in [0, \infty)$. There is a time-invariant state $\theta \in \{0, 1\}$, which denotes whether a particular story is true ($\theta = 1$) or false ($\theta = 0$). All players are endowed with a common prior $p_0 \equiv Pr(\theta = 1) \in (0, 1)$.

Each firm privately learns about the state by means of a one-sided Poisson signal: if $\theta = 1$, a signal confirming this arrives to each firm at a Poisson rate $\lambda > 0$. This learning process can be interpreted as one in which firms research a story by seeking confirmation that it is true. I assume such a learning process because is a reasonable approximation for the learning that takes in a breaking news setting. For instance, a firm researching whether an individual is the perpetrator of a terrorist attack can confirm this by discovering that they are either in custody or the subject of a search effort, but would find it much more difficult to prove their innocence while the investigation is still ongoing. To formalize this learning process, let $s_i \in [0, \infty]$ denote the time at which such a conclusive signal arrives to firm i, with $s_i = \infty$ denoting that a signal never arrives. I assume that $s_i \sim (1 - e^{-\lambda s_i})$ if $\theta = 1$, and $s_i = \infty$ if $\theta = 0$. I further assume that conditional on $\theta = 1$, s_i is i.i.d. across firms.

Each firm has a single opportunity to make a report over the course of the game. Notably, the firm does not choose what to report, but instead whether and when to do so. As the payoff function will soon illustrate, the content of this report can be interpreted as an assertion that the story is true, i.e., that $\theta = 1$. A report history H is a set $\{(i, t_i)\}_{i \in \{1, \ldots, N\}}$, pairing each firm i to the time at which it reported, t_i, with $t_i = \emptyset$ if the firm has not yet reported. Report histories are public information: at every time t, all players observe the current report history. Thus, firms not only learn about θ via their private signal, but also observationally by means of their rival firms' reports.

A firm who never reports earns a payoff of 0. Meanwhile, a firm who does report earns

$$k_n \alpha - \beta \mathbb{I}(\theta = 0).$$

Let us discuss the components of this payoff function. The first term, $k_n\alpha$, captures the firm's immediate payoff from making a report. This component captures the market share (i.e., viewership or readership) that the firm enjoys from reporting a story, and is the product of two separate components: k_n and α. While k_n captures the role of the firm's order n in its payoff, α denotes the *credibility* of the firm's report.

Let us now formally define k_n and α. A firm order of n denotes that the firm was the nth firm to report. I assume that the k_n are constants, where $k_1 \geq k_2 \geq ... \geq k_N \geq 0$. This assumption accounts for a key feature of the breaking news setting: competition. All else equal, firms who report early compared to their competitors enjoy greater market share. The firm's payoff is also increasing in the *credibility* of its report, α. A report's credibility is the consumer's belief, at the time the report was made, that the firm has received evidence that $\theta = 1$. Formally, this is the belief that $s_i \in [0, t]$, where t is the time of the firm's report. While the k_n are exogenous parameters, α is *endogenous*. In assuming this functional form, I assume that a firm's report will benefit it insofar that consumers believe it was informed. This captures the notion that consumers value accuracy in journalism, and thus only consume news to the extent that they find it credible.

The second term of the payoff function, $-\beta\mathbb{I}(\theta = 0)$, captures the ex-post penalty of error: a firm who reports when $\theta = 0$ incurs a penalty, given by a constant $\beta > 0$. This penalty captures the reputational harm a firm suffers from making a report that is later uncovered to be false.

2.1. Equilibrium

For each belief $p \equiv Pr(\theta = 1)$ and order $n \in \{1, ..., N\}$ of the next firm to report, let $F_{p,n}$ denote a distribution over future report times: at each (p, n), let t denote the span of time the firm waits before reporting conditional on not receiving a conclusive signal. Then, t is distributed according to $F_{p,n} \in \Delta[0, \infty]$, where $t = \infty$ denotes a lack of report altogether. A *Markov strategy* F is a collection of the $F_{p,n}$. I restrict attention to symmetric equilibria, and thus will omit the firm's index from the $F_{p,n}$ in much of the analysis below.

I place some restrictions on F. First, I assume that for all (p, n), $F_{p,n}$ must be piecewise twice differentiable and right-differentiable everywhere on $[0, \infty)$. This restriction grants analytical convenience and ensures that all equilibrium objects are well-defined.

Second, I impose a selection criterion (SC): a firm immediately reports once it has learned the story is true. This criterion is stated formally as follows:

Definition 1. F satisfies (SC) if

$$F_{1,n}(t) = 1 \text{ for all } t \geq 0, n \in \{1, ..., N\}.$$

(SC) imposes that firms do not abstain from reporting a story they know to be true. It serves the purpose of ruling out unintuitive equilibria with periods of silence, which can only be supported by off-path beliefs that reports made during these gaps entail little or no credibility. An implication of this assumption is that fixing any starting belief p, all players who have not yet reported will share the same *common belief* about the state after t time has passed. I denote this common belief by $p(t)$.

Finally, while defining strategies in this way, i.e. with a separate distribution for each (p, n), is convenient, it introduces redudancy. Thus, I must impose a consistency condition to ensure that the $F_{p,n}$ are consistent with each other whenever on-path.[4] This condition stipulates that $F_{p,n}$ and $F_{p(t),n}$ are related via the following formula:

$$F_{p(t),n}(s) = \frac{F_{p,n}(s+t) - F_{p,n}(t_-)}{1 - F_{p,n}(t_-)} \text{ for all } s \geq 0 \text{ whenever } F_{p,n}(t) < 1, \tag{1}$$

where $F_{p,n}(t_-) \equiv \lim_{\tau \uparrow t} F_{p,n}(\tau)$. This formula is an immediate result of Bayes Rule.

Before proceeding, I define two intuitive terms to describe reporting behavior: *faking* and *truth telling*. A report is *fake* if it is made by a firm despite lacking independent confirmation, i.e., a signal $s^i \neq \emptyset$. Meanwhile, a report that is made in response to such a signal is *truthful*. I use these terms to not only describe a firm's report, but also its behavior: a firm is faking if it is sending a fake reports, while it is truth telling if its reports are exclusively truthful. Given the above selection assumption, strategies only differ in the distributions they place over fake reports.

I seek a symmetric perfect Bayesian equilibrium of this game. This is defined as a Markov strategy F paired with beliefs α and p at each history such that F satisfies sequential rationality and both α and p are consistent with Bayes Rule.

The consistency of α with Bayes Rule implies that it must be given by the following formula at all (p, n) on-path: [5]

[4] This condition is analogous to the closed-loop property specified in Fudenberg and Tirole (1985). I adopt the term *consistency condition* from Laraki, Solan, and Vieille (2005), who define this condition for a general class of continuous-time games of timing.

[5] Formally, the formula is derived by applying Bayes Rule to a discrete-time approximation of the beliefs that obtain under this game. This derivation is presented in Appendix A.

$$\alpha_n(p) = \begin{cases} \frac{\lambda p}{\lambda p + b_n(p)} & \text{if } F_{p,n}(0) = 0 \\ 0 & \text{if } F_{p,n}(0) > 0 \end{cases} \quad (2)$$

where $b_n(p) \equiv F'_{p,n}(0+)$ denotes the right-derivative of $F_{p,n}$ at 0. That is $b_n(p)$ denotes the *instantaneous hazard rate* of fake reports by a firm. This can be interpreted as the intensity with which a firm fakes at a particular (p, n).

This formula is intuitive. If $F_{p,n}(0) > 0$, there exists a point mass of reports at (p, n). However, because conclusive signals are continuously distributed over time, the probability with which a valid report is made at (p, n) is zero. Thus, the consumer and all competing firms know with certainty that a report made at (p, n) was fake, and thus assigns to it a credibility of zero. Meanwhile if there does not exist a point mass of reports at (p, n), credibility is assessed by comparing the instantaneous arrival rate of truthful reports (λp) to that of fake reports ($b_n(p)$), assigning higher credibility to reports made when the hazard rate of fake reports is comparatively low.

3. Equilibrium characterization

3.1. Properties of equilibrium

I begin by establishing two necessary conditions on the firm's equilibrium strategy that will guide the equilibrium characterization. Namely, I show that there are *no jumps* and *no gaps* in the distribution of fake reports whenever credibility is less-than-perfect. These two properties arise in other games with continuous strategy spaces, albeit in different forms.[6] In my setting, these properties hold even in the absence of competition. As I will illustrate below, this is because they are driven by the endogeneity of the firm's payoff.

These two properties are stated formally as Lemma 1:

Lemma 1. *In equilibrium, at any (p, n) on-path $F_{p,n}$ is*

(a) continuous at all t whenever $p < 1$

(b) strictly increasing at any t such that $\alpha_n(p(t)) < 1$.

Let us begin by considering part (a) of Lemma 1, i.e., the "no jumps" property. This states that fake reports are distributed continuously over time whenever a firm is not certain that the story is true. I.e., there can never be a point mass in faking when $p < 1$. Notably, this

[6] In particular, similar properties have been established in war of attrition games (Hendricks, Weiss, and Wilson (1988)) and all-pay auctions (Baye, Kovenock, and De Vries (1996)).

property holds even when competition is absent ($n = 1$). I will now argue that such point masses cannot occur because they give rise to a profitable deviation. This is driven by the association between the firm's strategy and its credibility in equilibrium (i.e, (2)): reports that are made whenever there is a point mass in faking yield zero credibility. Meanwhile, faking while also not being certain than the story is true yields a strictly positive expected penalty $\beta(1-p)$. This implies that a firm's value from faking at such a time is strictly negative. Thus, the firm can profitably deviate by truth telling at that time: truth telling precludes the firm from making an error, and therefore ensures a weakly positive payoff.

Next, let us turn to part (b) of Lemma 1, the "no gaps" property. This states that whenever the firm is less-than-fully credible, the hazard rate of fake reports must be strictly positive. In other words, firms must mix between faking at all times in which $\alpha_n(p(t)) < 1$. Again, this property results directly from the formula for α (2): whenever credibility is less-than-perfect, the firm must be faking at some positive rate $b_n(p)$. While straightforward, this property of the firm's strategy has important implications for incentives in equilibrium. In particular, it implies that whenever a firm's credibility is less-than-perfect, it must be indifferent between faking instantly and waiting an infinitesimal increment of time before faking. This indifference condition will be crucial to characterizing the firm's behavior in equilibrium.

3.2. The monopoly benchmark and role of commitment

Before proceeding with the full model characterization, let us consider the special case in which there is a single firm, i.e. $n = 1$. This serves two purposes. First, it elucidates the forces at play when competition is absent. In particular, it shows that errors can occur even without competition, and that such errors are driven by a lack of commitment power by the firm. Second, it serves as a benchmark for understanding the marginal impact competition on firm incentives and behavior.

I now state the monopoly characterization in terms of the firm's credibility. Because there is a single firm, I will drop the n index from all functions and parameters.

Claim 1. *Under a monopoly, for all p on-path*

$$\alpha(p) = \min\{\beta/k, 1\}.$$

Claim 1 establishes two core facts about the monopoly equilibrium. First, credibility is constant over time. As I will illustrate below, constant credibility implies that the firm's reporting behavior is often *not* static. Namely the firm becomes more truthful over time. Second, the monopolist's credibility is weakly increasing in β, and is less-than-perfect whenever β is sufficiently small. That is, errors occur even without competition, whenever the ex-post penalty from erring is sufficiently small. The remainder of this subsection is dedicated to understanding why these two properties hold under a monopoly, and what they imply about the firm's reporting behavior.

Let us begin by understanding why credibility must be constant in equilibrium. Recall that a firm mixes between faking at all times in which its credibility is less-than-perfect (Lemma 1(b)). Thus, whenever $\alpha_n(p) < 1$, the firm must find it optimal to both fake immediately and after some short wait dt.[7] By the martingale property of firm's belief p about the state, both of these strategies will yield the same expected penalty from error $\beta(1-p)$. Then, in order to ensure that both strategies are optimal, the firm's *prize* from reporting must be the same as well. I.e., credibility must be constant. What is implicit in this reasoning is that waiting is not costly to a monopolist. In part this is because waiting is not intrisically costly to the firm, i.e., future payoffs are not discounted. This is also due to the fact that a monopolist does not face competitors, and thus does not incur the *implicit* cost to waiting that comes from being preempted. In fact, we will later illustrate that a cost of preemption precludes α from being constant to equilibrium (Subsection 4.1).

The constant nature of the monopolist's credibility implies that its reporting behavior will often not be static: the hazard rate of faking (b) strictly decreases over time and tends to zero whenever credibility is less-than-perfect. That is, even when a firm fakes, it will become gradually more truthful over time. This is illustrated by Figure 1, which graphs b over time. While the decreasing nature of b follows directly from (2), there is also an intuition behind this. The more time passes without observing a report, the more skeptical the consumer becomes that the story is true. This declining belief an artifact of the firm's one-sided Poisson learning process: the absence of a report means that the firm has not received a conclusive signal, and thus the common belief $p(t)$ that the story is true decays over time.[8] This means that the consumer believes that truthful reports will become increasingly less probable. To ensure that the firm's credibility remains constant, the hazard rate of fake reports must decline as well, and eventually vanish.

Let us now consider the second property mentioned above, i.e., that a monopolist will err with positive probability as long as the penalty of error is sufficiently small. This property demonstrates that competition alone is not responsible for errors in equilibrium. I now

[7] Implicitly, this relies on the assumption that $\alpha_n(p(t))$ is continuous over time: this ensures that if $\alpha_n(p) < 1$, then $\alpha_n(p(dt)) < 1$ for dt sufficiently small. While I do not discuss this here, I formally establish continuity in Appendix D (see Lemma 4).

[8] Formally, in the monopoly case, $p(t) = \frac{pe^{-\lambda t}}{pe^{-\lambda t}+(1-p)}$.

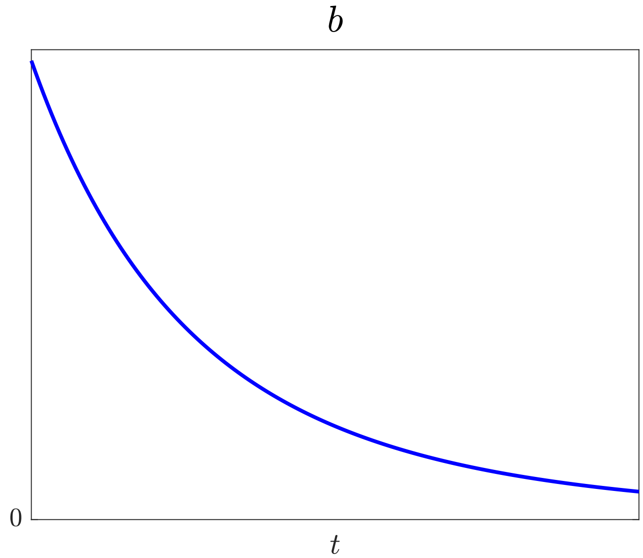

Figure 1: Hazard rate of fake reports in monopoly case when $\beta < k_n$.

argue that such errors are driven by a firm's inability to *commit* to a reporting strategy.

To illustrate this point, let us first understand why truth telling cannot be sustained when $k > \beta$. A firm that truth tells in equilibrium enjoys full credibility when making a report. Thus, the firm's payoff from reporting when the story is false ($\theta = 0$) is positive: the immediate payoff of the report, k, strictly exceeds the penalty β from error. Consequently, faking is a profitable deviation. When $\theta = 1$, both faking and truth telling will ensure the firm reports eventually, earning a payoff of k. However, faking is strictly better for the firm when $\theta = 0$: it ensures a strictly positive payoff whereas truth telling yields nothing. The profitability of this deviation is driven by the fact that consumers cannot discern by merely observing a report whether it is fake. They only hold a belief about this, i.e., they assess credibility. While this assessment is made rationally based the beliefs about the firm's strategy, the firm can always deviate after credibility has been determined. This is because the firm is unable *commit* to a reporting strategy, i.e., to forbid itself from deviating after the credibility has been assessed. Faking is especially tempting to the firm after its credibility has been assessed because it will not damage the firm's immediate payoff of reporting.

Let us now consider how a monopolist firm would behave if it did have the ability to commit. That is, suppose that the the firm could announce its strategy at the start of the game, and was unable to deviate from it once credibility had been assessed.[9] Under commitment, faking is more costly for the firm: it would always damage the firm's credibility,

[9] While we discuss the commitment solution informally here, a formal treatment is presented in Appendix F.

and thus its immediate payoff from reporting.

One can immediately see that under commitment, the firm would always choose truth telling over its non-commitment strategy even when $\beta < k$. By committing to truth telling, the firm guarantees that it will earn a payoff of k if $\theta = 1$, and 0 if $\theta = 0$. Meanwhile, under the no-commitment equilibrium, the firm will earn strictly less (β) when $\theta = 1$, because it's credibility is strictly lower. Meanwhile, it will also earn 0 when $\theta = 0$: though the firm may fake, its payoff from faking is exactly equal to the penalty of error, meaning that the firm will break even. In fact, one can show that truth telling is not only better than the equilibrium strategy, but that it is the unique commitment solution under a monopoly. That is, given the ability to commit, a monopolist would never commit errors. This result is presented formally in Appendix F. We can thus conclude that a lack of commitment is responsible for errors under a monopoly. This also illustrates an important point about a firm's incentives: while commitment makes faking more costly to the firm, it in fact leaves the firm *better off* in equilibrium. This observation points to a larger theme that will persist even under competition: firms fake not because it intrinsically benefits them, but because it is a side effect of their strategic considerations.

3.3. Full model characterization

Here, I establish existence and uniqueness of an equilibrium in the full model. To this end, I show that any equilibrium is the solution to a recursive set of boundary value problems. Specifically, whenever the firm is not truthful, its credibility must satisfy an ODE and appropriate boundary condition. Characterizing the equilibrium in this way not only allows one to establish existence and uniqueness, but lays the foundation for the economic analysis that follows.

I begin by establishing the precise conditions under which a firm is truthful. I present this result for two reasons. First, it serves as a first step towards a full characterization. Second, while I illustrate this point more generally in the section that follows, this result shows how competition can deteriorate credibility and exacerbate faking.

Proposition 1. *In equilibrium, at any (p, n) on-path, $\alpha_n(p) = 1$ if and only if the following two conditions hold:*

1. $k_n \leq \beta$
2. $p \leq p_n^* \equiv \min\{\frac{k_n - \beta}{\frac{k_n}{N-n+1} - \beta}, 1\}$.

This result provides two conditions, on the model parameters and the common belief about the state, that are both necessary and sufficient for the firm to truth tell. The first condition, that $k_n \leq \beta$, was both necessary and sufficient for truth telling under a monopoly (Claim 1). However, Proposition 1 asserts that when firms face competition, this condition alone is not enough to ensure truth telling. A second condition is required: the common belief about the state must be sufficently low, lying below some threshold p_n^*. That is, firms must also be sufficiently skeptical about the validity of the story.

The necessity of this second condition illustrates an important point: *truth telling is harder to sustain under competition*. To understand why, note that truth telling is possible only if the firm does not have an incentive to deviate by faking. In the monopoly case, this was true as long as the cost of an error (β) outweighed the benefit from reporting (k). However, competition introduces an additional cost to truth telling: the risk of being preempted. If a firm engages in truth telling, there is a risk that its opponent learns the story is true, and thus reports first. A firm can evade this risk by faking, which ensures that it will not be preempted.

In the above reasoning, we took for granted that being preempted is costly for the firm whenever it is truth telling. Let us now explain why this is true. It is most obvious in the *winner takes all* case: all firms, with the exception of the first to report, are guaranteed to earn a payoff of zero, i.e., $k_n = 0$ for all $n > 1$. In this case, the costliness of being preempted is an artifact of the model parameters: a firm who is preempted will earn nothing from reporting. Generally, however, the decreasing nature of the k_n alone does not guarantee that being preempted is costly: improved credibility for succeeding firms could endogenously counteract the decay in the k_n and make being preempted costless, or even beneficial. However, one can show that being preempted must be costly for the firm whenever it is truth telling. This is due to the fact that truthfulness guarantees that the firm enjoys full credibility, which leaves no room for improvement in credibility.

Now, let us consider the significance of the second condition, i.e., that the firm will only be truthful if it is sufficiently pessimistic about the story's validity. While truth telling entails a risk of being preempted, faking entails a different kind of risk: that of making an error and incurring penalty β. Both of these risks depend on the belief p about the state. A higher belief p is associated with a lower risk of error and a higher risk of being preempted, both of which make faking relatively more appealing to the firm, and thus, make truth telling more difficult to sustain. While it is immediate that a greater p implies a smaller risk of error, that it implies a greater risk of being preempted is less obvious. To see why this is true, note that if the story is true, a competitor may preempt for two different reasons: it has confirmed the story, or is faking. However, if the story is false, preemption is triggered solely by faking, and thus the risk of being preempted is lower. Thus, a firm who is more confident in the story will perceive its risk of being preempted to be greater.

While Proposition 1 pins down the conditions under which the firm is fully credible in equilibrium, it remains to characterize the firm's behavior when truth telling does not hold. To this end, we obtain a key result: the firm is faking, credibility must satisfy a particular ODE and limit condition.

Proposition 2. *In equilibrium, at all (p, n) on-path where $k_n \geq \beta$ or $p > p_n^* \equiv \frac{k_n - \beta}{k_n/n - \beta}$, the following ODE must be satisfied:*

$$\alpha_n'(p) = -\frac{1}{k_n(1-p)\alpha_n(p)} \frac{N-n}{N-n+1} [k_n \alpha_n(p) - V_{\tilde{p},n+1} - \beta(1-\alpha_n(p))(1-p)] \qquad \text{(ODE)}$$

where $\tilde{p} \equiv \alpha_n(p) + (1-\alpha_n(p))p$.

In addition, $\lim_{p \to 0+} \alpha_n(p) = \beta/k_n$ must hold if $k_n > \beta$, and $\lim_{p \to p_n^*+} \alpha_n(p) = 1$ if $k_n \leq \beta$.

The proof for Proposition 2 relies critically on our above observation that whenever a firm is less-than-fully credible, it must mix between faking immediately and faking after some short wait, and thus must be indifferent between the two. To state this formally, let δ_s denote the pure strategy distribution that places full mass on faking after s time has passed. In particular, δ_0 denotes immediate faking, while δ_{dt} denotes faking after some short wait $dt > 0$. The indifference condition can then be written as follows:

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_{dt})$$

where $V_{p,n}(\cdot)$ denotes the firm's value from playing a particular strategy at (p, n).

To see how this indifference condition implies (ODE), note that a Taylor approximation of the firm's value from waiting, $V_{p,n}(\delta_{dt})$, yields the following:

$$V_{p,n}(\delta_{dt}) - V_{p,n}(\delta_0) = [\frac{dp}{dt}(k_n \alpha_n'(p)) - \frac{\lambda p(N-n)}{\alpha_n(p)}(V_{\tilde{p},n} - V_{\tilde{p},n+1})]dt + o(dt^2) \qquad (3)$$

(3) is intuitive. It tells us that waiting to fake, rather than faking immediately, has two implications for the firm's payoff. The first is that the firm's credibility $\alpha_n(p)$, and thus the payoff enjoyed from reporting, may potentially change. This change in credibility is approximated by $\frac{dp}{dt}(k_n \alpha_n'(p))dt$. In addition, by waiting, the firm risks being *preempted*. Precisely, with probability $\frac{\lambda p(N-n)}{\alpha_n(p)}dt$ the firm is preempted, in which case its expected payoff will decline by $V_{\tilde{p},n} - V_{\tilde{p},n+1}$. We interpret this decrease in value as the firm's *cost from being preempted*.

Let us now examine both the probability and cost of preemption more closely. As one might expect, the probability of being preempted is increasing in the number of rival firms

$(N - n)$ and the expected rate at which these rivals are able to confirm the story (λp). It is also decreasing in equilibrium credibility. This is due to the fact that lower credibility firms are more likely to fake, and thus pose a greater preemptive threat.

As for the firm's cost of being preempted, let us begin by considering the second component of this expression, given by $V_{\tilde{p},n+1}$. This denotes the firm's continuation value in the event that it is preempted. Importantly, being preempted not only affects the firm's order but also the the common belief about the state. While the common belief was p prior to the rival firm's report, it increases to $\tilde{p} \equiv \alpha_n(p) + (1 - \alpha_n(p))p$ following the report. This is due to observational learning. Specifically, a rival firm's report means one of two things: either the report was triggered by the arrival of a conclusive signal, in which case the story is certainly true and the belief would become p, or it was not, in which case the report provides no new information and the belief remains p. The common belief following this report is a weighted sum of these two conditional beliefs. In particular, the weight given to the rival firm's report being informed by a conclusive signal is precisely its credibility at the time of the report, $\alpha_n(p)$. This new common belief will in turn determine the firm's continuation value in the event that it is preempted.

The cost of being preempted measures the impact of being preempted on the firm's continuation value. I.e., it measures how much the firm's continuation value from being preempted differs from that in which it is not. Importantly, both continuation values are assessed at the common belief after being preempted, $\tilde{p}$. In this sense, we can view the cost of being preempted as the firm's *ex-post regret* from being preempted.

In order for the indifference condition to be satisfied, the linear term of (3) must equal zero. This equality yields (ODE). That is, $\alpha_n(p(t))$ must change in precisely such a way that preserves the firm's indifference condition.

In addition to establishing (ODE), Proposition 2 imposes a limit condition on the firm's credibility. This limit condition always applies at the boundary of the region of beliefs in which the firm is faking. Let us first consider the case where $k_n \leq \beta$. Recall from Proposition 1 that in this case, $\alpha_n(p) = 1$ whenever $p \leq p_n^*$. We must then have that $\alpha_n(p)$ limits to 1 as the belief approaches p_n^*. If it did not, then as the belief approached p_n^*, the firm could profitably deviate by not faking immediately, and rather waiting until p_n^* is reached to do so. Thus, this limit condition is needed to sustain the firm's indifference condition.

Let us next consider the case where $k_n > \beta$. In this case, the firm never truth tells in equilibrium, and thus the indifference condition must always be satisifed. As the common belief p approaches zero, a firm who fakes does so being increasingly certain that its report is erroneous, and will incur penalty β. Thus, the firm's payoff from faking limits to the following:

$$\lim_{p \to 0+} V_{p,n}(\delta_0) = k_n \lim_{p \to 0+} \alpha_n(p) - \beta$$

Separately, even though the firm sometimes fakes, it must also *never fake*, i.e., play strategy δ_∞, with positive probability. This guarantees that the hazard rate of fake reporting remains low enough to ensure that credibility remains sufficiently high, and thus that the firm will indeed find it optimal to fake.[10] As $p \to 0+$, the value of truth telling tends to zero, as it becomes increasingly likely that the firm never reports. We can thus see that the limit condition in this case, $\lim_{p \to 0+} \alpha_n(p) = \beta/k_n$, is precisely what is needed to ensure that the firm is indifferent between faking and truth telling.

To take stock, Proposition 1 and Proposition 2 provide two different necessary conditions on equilibrium credibility. They establish a region under which truth telling must occur in equilibrium (Proposition 1), and show that otherwise, credibility must satisfy a recursive boundary value problem (Proposition 2). While I relegate its formal statment to the appendix, one can show these two conditions are not only necessary, but sufficient, for an equilibrium.[11] Establishing sufficiently entails showing that the firm cannot profitably deviate as long as its credibility satisfies these conditions. There is a clear intuition for this. On the region where truthfulness is necessary, it must also be optimal. This is for the same reason that faking cannot occur on this region: the expected cost of erring is so high that it counteracts any benefit that faking might bring. Meanwhile on the region where $\alpha_n(p) < 1$, the firm's strategy involves faking. Faking is optimal on this region because (ODE) guarantees it. In particular, it ensures firm's indifference condition holds, i.e., that faking at any such time is optimal.

I thus establish that the equilibrium is fully characterized by the solution to a recursive boundary value problem. While I do not have a closed-form solution to this problem on the region where $\alpha_n(p) < 1$, I am able to establish both existence and uniqueness of a solution. This is done by means of the Picard Theorem. This result is stated formally as Theorem 1.

Theorem 1. *There exists a unique equilibrium (where uniqueness applies at (p, n) on-path).*

4. Economic Implications

In this section, I consider key the economic implications of this equilibrium. In particular, I explore two notions: (1) the dynamics of firm behavior and (2) the impact of competition on both credibility and the prevalence of errors.

[10] I formalize this result in the Appendix as Lemma 2.

[11] This result is stated in the Appendix as Lemma 5.

4.1. Equilibrium dynamics

Let us consider how a firm's credibility and reporting behavior evolves over the course of time. These results will not only illustrate when firms are most prone to erring, but will also allow us to better understand the endogenous nature of the firm's incentives.

Dynamics take two separate forms in equilibrium: *continuous* changes and *discrete* changes. As I will show, continuous changes occur in the absence of any new reports, while discrete changes are triggered by a new report. More formally, let us denote a *subgame* by a pair (p, n), where p denotes a starting belief and n the order of the next firm to report. I claim that fixing a subgame, i.e., assuming that no new reports are made, the firm's credibility will change continuously over time. In particular it will *gradually improve* whenever preemptive concerns are present. This result is stated formally as Proposition 3.

Proposition 3. *For all (p, n) on-path, $\alpha_n(p(t))$ is weakly increasing in t. Furthermore,*

1. *If $\beta > k_N$, then $\alpha'_n(p(t)) > 0$ whenever $\alpha_n(p(t)) < 1$.*
2. *If $\beta \leq k_N$, then $\alpha_n(p(t))$ is constant in t.*

While $\alpha_n(p(t))$ must be constant under a monopoly, competition can introduce dynamics. Proposition 3 asserts that as long as $\alpha_n(p(t))$ has not reached its upper bound of 1, it is strictly increasing precisely when being preempted is costly to the firm.

Formally, this follows from (ODE). It is especially clear when we write (ODE) in the following form:

$$\frac{d}{dt}\alpha_n(p(t)) = \frac{\lambda p(N-n)}{\alpha_n(p(t))k_n}[V_{\tilde{p},n} - V_{\tilde{p},n+1}]$$

We can see that $\alpha_n(p(t))$ must strictly increase over time whenever the cost of preemption is strictly positive.

There is also a clear intuition for this result. Whenever the firm is less-than-fully credible, it must be indifferent between faking immediately and waiting some period of time before doing so. However, if credibility remained constant, this indifference would fail whenever preemption is costly: the firm would obtain the same expected payoff from reporting in both cases, but by reporting immediately would avert being preempted. To ensure that indifference is preserved, the firm must somehow be compensated for waiting. This can only be achieved by means of strictly increasing credibility. While waiting presents a cost to being preempted, a strictly increasing $\alpha_n(p(t))$ ensures that the firm's report will be rewarded more in the event that it is not preempted. Thus, the increasing nature of $\alpha_n(p(t))$ is crucial to balancing the firm's equilibrium incentives: it endogenously mitigates the haste-inducing effects of preemptive risk.

Let us now consider the implications of this result. It asserts that news reports that are made with greater delay for research are generally more trustworthy in the eyes of consumers. That is, all else equal, consumers will have greater trust in a firm's journalistic standards when a report is not made quickly. In this sense, this result conforms with consumers' stated concerns about hasty reporting. This model provides a justification for such concerns that are grounded in the firm's incentives. Furthermore, by the same reasoning presented in the monopoly section, the increasing nature of credibility within a subgame implies that firms become gradually more truthful over the course of the game. That is, $b_n(p(t))$ is strictly decreases over time whenever the firm is not fully credible.

Finally, while Proposition 3 asserts that $\alpha_n(p(t))$ must be strictly increasing when there is a cost to being preempted, this is not always the case. Specifically, when $k_N \geq \beta$, being preempted is costless in equilibrium (i.e., $V_{\tilde{p},n} - V_{\tilde{p},n+1} = 0$), and thus $\alpha_n(p(t))$ is constant. In other words, preemptive concerns endogenously disappear whenever the ex-post cost of error, β, is sufficiently small. Formally, the credibility function will adjust in such a way that ensures $k_n \alpha_n(p) = k_{n+1}\alpha_{n+1}(p)$ for all p. This highlights an notable feature of our model: competition alone does not imply preemptive concerns. Even if competition is present, credibility can change in such a way that makes preemption costless.

Let us now consider discrete changes in the firm's credibility and faking. While credibility changes continuously within a subgame, a rival report will cause the firm's subgame to change. That is, a report made at (p, n) will cause the order of the next reporter to increase to $n + 1$ and the common belief to increase to $\tilde{p}$. This will in turn result in discrete jumps in the firm's credibility and hazard rate of faking (b). These discrete jumps are apparent in Figure 2, which plots a simulation of α and b over the course of the game. As these graphs illustrate, jumps in both α and b are not monotonic. An opponent report may trigger either a boost or decline in α and b. This can be seen in Figure 2, while the first four reports cause credibility to decrease and faking to increase, the fifth report causes credibility to decrease and faking to increase.

These first four reports illustrate a *copycat effect*, in which one firm's report causes an immediate a surge in the rate at which others fake.

Let us now consider what is responsible for this copycat effect. To do so, first note that the discrete change in credibility that happens when a firm makes the nth report under common belief p is given by the following:

$$\alpha_{n+1}(\tilde{p}) - \alpha_n(p)$$

where again $\tilde{p} > p$ denotes the common belief in the immediate aftermath of the report.

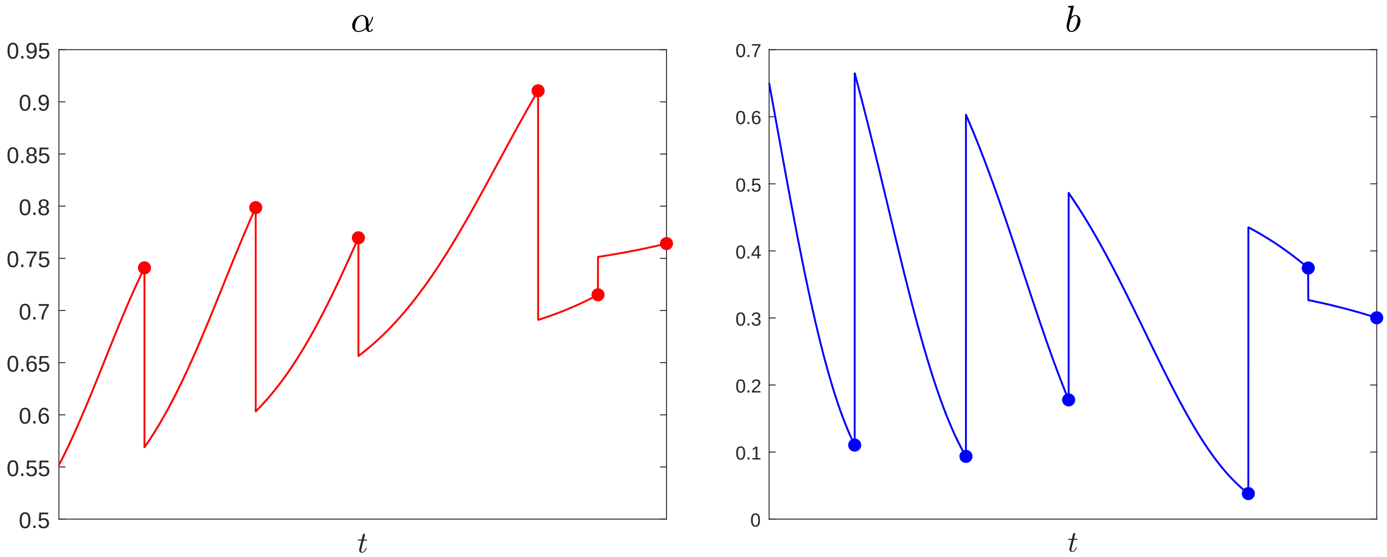

Figure 2: Simulations of crediblity and the hazard rate of fake reports, respectively, over the course of the game. Discrete jumps in both graphs signify that a firm has made a report.

This expression shows that a report by one firm affects credibility by imposing two different changes to the environment. First, it impacts the order of the next firm, i.e., by ensuring that the next firm to report will be the $n+1$th firm to report, rather than the nth. Second, it causes a discrete upwards jump in the common belief: firms will learn observationally from the report of their opponent, and thus become more confident that the story is true. The following decomposition isolates the respective impacts of these two changes:

$$\alpha_{n+1}(\tilde{p}) - \alpha_n(p) = \underbrace{[\alpha_{n+1}(\tilde{p}) - \alpha_{n+1}(p)]}_{\text{change in belief}} + \underbrace{[\alpha_{n+1}(p) - \alpha_n(p)]}_{\text{change in order}}$$

The effects of a change in order alone, $\alpha_{n+1}(p) - \alpha_n(p)$, can have an ambiguous impact on firms' credibility in equilibrium. In particular, it hinges on the way in which the maximal prize k_n changes with a firm's order.

However, the effect observational learning is not ambiguous: it will always cause a deterioration in credibility. Formally, $\alpha_{n+1}(\tilde{p}) - \alpha_{n+1}(p)$ will always be negative in equilibrium, and strictly so whenever premeptive concerns are present (i.e., whenever $k_N < \beta$). The negative correlation between credibility and the firm's belief that the story is true is also apparent in Proposition 3: later reports are associated with lower common beliefs about the story being true, and also with higher credibility. There is also a clear intuition to this: the more pessimistic the firm is about the story's validity, the higher its expected penalty from faking will be. This in turn yields the firm less willing to fake, and thus more credible

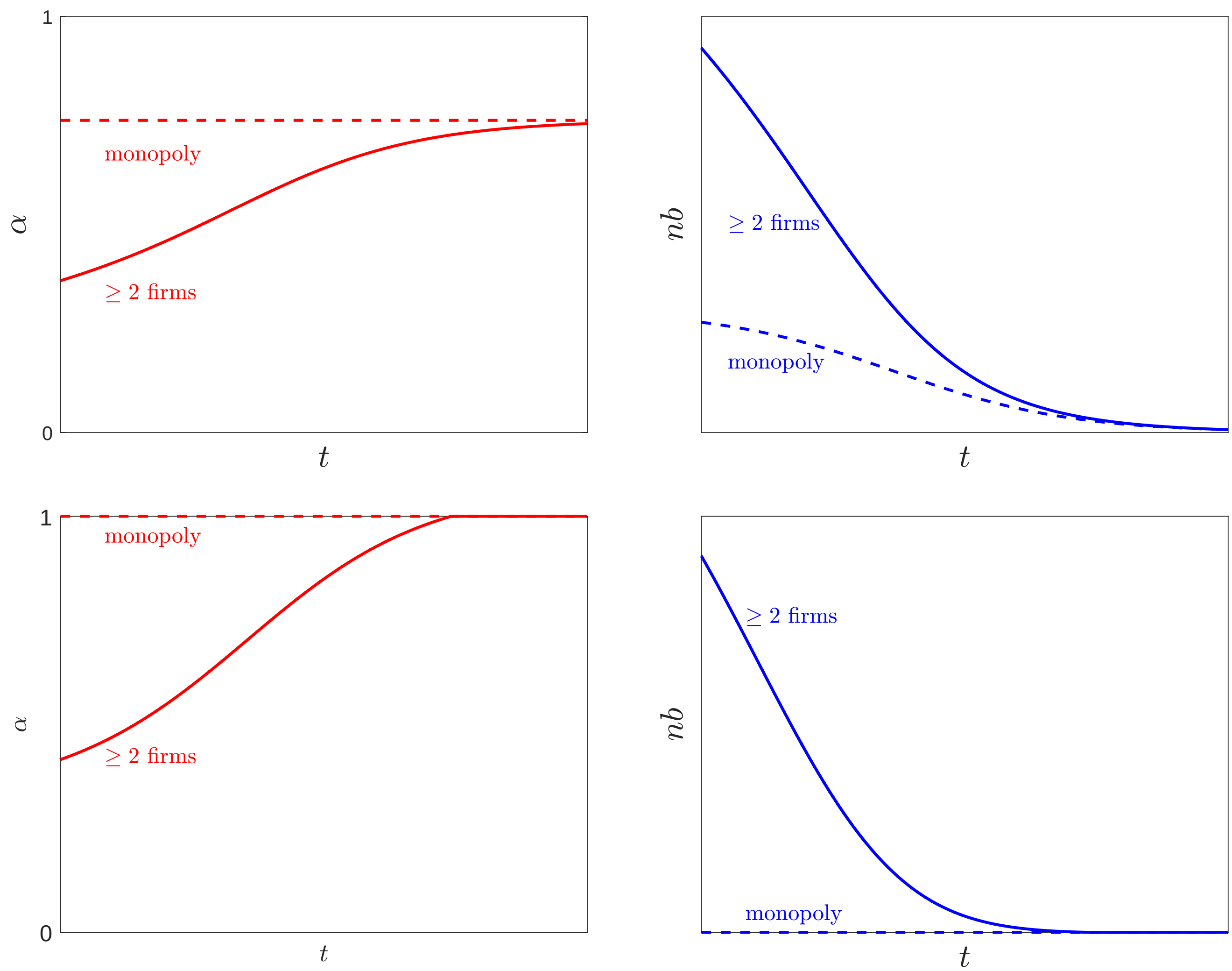

Figure 3: Credibility $\alpha_n(p(t))$ (left) and the hazard rate of faking in the market $nb_n(p(t))$ (right) under competition and a monopoly. Top row depicts case where $k_n > \beta$, while bottom row depicts case where $\beta \in (k_N, k_n)$.

in equilibrium. We thus see that the downwards jumps in credibility are caused, at least in part, by observational learning.

4.2. Effects of Competition

In this section, I consider the impact of competition on both credibility and faking in equilibrium. I assess the impact of competition by comparing the equilibrium under competition ($n \geq 2$) to that under the monopoly benchmark.

In order to isolate the effects of competition, I assume that the total ability of the market to learn is constant across these two cases. In particular, I assume that if each firm has ability λ under competition, then the firm has ability $n\lambda$ under the monopoly benchmark. In making this normalization, one ensures that our comparison accounts for only the impact of competition per se and does not confound this with the effects of an increased aggregate ability to learn that firm entry may entail. I do however consider the effects of market entry in the comparative statics section below, in which we do not normalize the total ability to learn.

These findings are shown in Figure 3, which depicts both credibility and the hazard rate of faking in the market ($nb_n(p(t))$) within a subgame, i.e., fixing a p and an n. The top and bottom row show the case where $\beta \in (k_N, k_n)$ and $\beta > k_n$, respectively. In both cases, we see that *competition causes a deterioration in credibility and an increase in faking*. The effect of competition in this case is driven by the cost of preemption that it induces. Firms are more inclined to fake, and thus less credible because the cost of preemption makes truth telling more costly. When $\beta \in (k_N, k_n)$, faking occurs even under a monopoly, but moreso under competition. That being said, the effects of competition dissipate over time, as the competition level of credibility limits to the monopoly level as time passes. Meanwhile, in the case where $\beta > k_n$, a monopolist firm will *never fake*, faking does temporarily occur under competition. Again, the effects of competition are greatest early on with firms faking gradually less as time passes.

5. Comparative Statics

In this section, I consider how the equilibrium changes with the parameters of the model. This will shed light on how various features of the news market can either exacerbate or curb erroneous reporting. These findings are stated as Proposition 4.

Proposition 4. *In any equilibrium, for any n, $\alpha_n(p(t))$ is*

(a) weakly increasing in β, and strictly so whenever $\alpha_n(p(t)) < 1$.

(b) weakly increasing in λ, and strictly so for $t > 0$ whenever $\alpha_n(p(t)) < 1$ and $k_N < \beta$.

(c) weakly decreasing in N, and strictly so whenever $\alpha_n(p(t)) < 1$, when $t \in [0, \bar{t}]$ for some $\bar{t} > 0$.

Part (a) states that no matter when a firm reports, it will be more credible under high β. This result is intuitive: a higher ex-post cost of error means firms are less likely to fake, and thus more credible. This result is a consequence of the firm's equilbrium incentives: a higher β makes faking more costly. This will either induce the firm to resort to truth telling instead, or require that it is compensated for this coster faking with greater credibility.

Now, let us consider the comparative static on λ. This result is also intuitive: it states that credibility is higher whenever firms have a greater ability to learn. Let us now understand

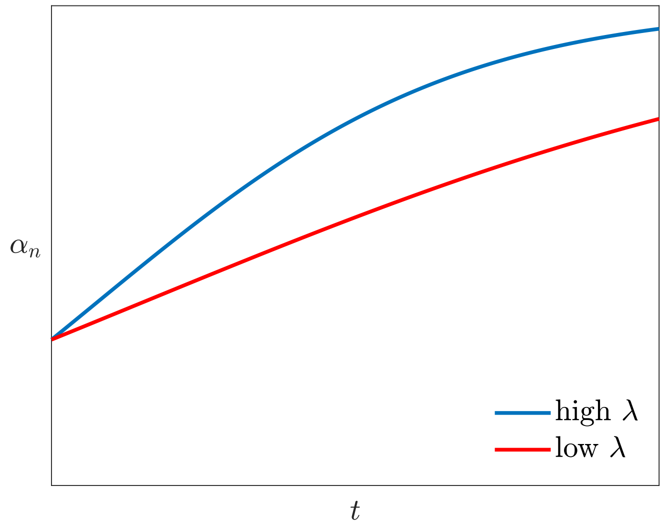

Figure 4: A simulation of $\alpha_n(p(t))$ when $\lambda = 1$ (blue line) and $\lambda = 0.5$ (red line). For the remaining parameter values, the following specifications were made: $\beta = 0.5$, $p_0 = 0.7$, $N = 8$, $k_n = 0.7^{(N-n)}$.

what is driving this result. We first note that at any belief p the firm may hold, a change in λ will have *no effect* on $\alpha_n(p)$ in equilibrium. This is due to the fact that λ does not enter the boundary value problem which dictates the firm's credibility, and thus changes in λ have no effect $\alpha_n(p)$. However, changes in λ will have an effect on the time path of the common belief $p(t)$. Under a higher λ, firms learn about the state more quickly, and thus $p(t)$, the belief that $\theta = 1$ conditional on no reports, will decay faster. That is, firms will be more pessimistic about the story's validity at any time $t > 0$ when λ is higher. This greater pessimism about the story translates to a higher expected cost of erring, which thus makes faking more costly. As was true of the comparative static on β, this increased cost of faking must be counterbalanced by a higher credibility $\alpha_1(p(t))$ at every time $t > 0$. This comparative static is illustrated by Figure 4, which shows simulations of the firm's credibility function under both high and low values of λ.

Let us finally consider the comparative static on the total number of firms, N. While it pertains to the level of competition, this exercise is notably distinct from our analysis in the previous section. Therein, we studied the overall impact of competition on equilibrium outcomes. This was done by comparing the case where competition is present ($N > 1$) to the monopoly case ($N = 1$) while holding constant the total learning ability of the market, $N\lambda$. With this comparative static, we are instead considering the *marginal* impact of an additional firm entering the market. In particular, we do not hold fixed the total learning ability of the market. Rather, I assume that this additional firm adds to the total learning

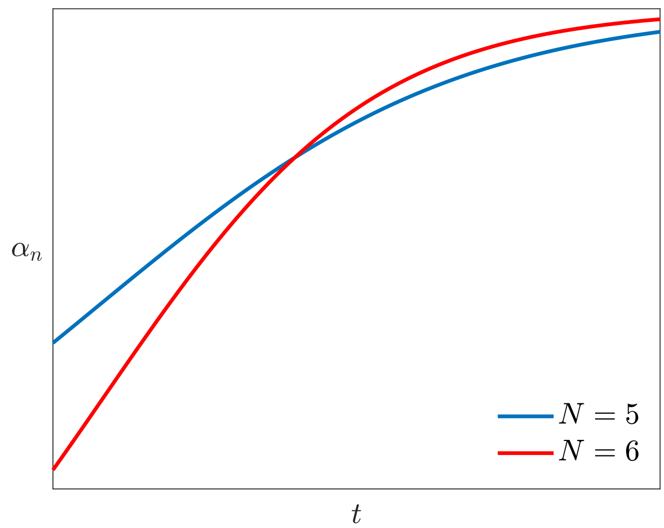

Figure 5: A simulation of $\alpha_n(p(t))$ when $N = 5$ (blue line) and $N = 6$ (red line). For the remaining parameter values, the following specifications were made: $\beta = 0.5$, $p_0 = 0.7$, $\lambda = 1$, $k_n = 0.7^{(N-n)}$.

ability of the market. In doing so, one can study the effect of *proliferation* in the news industry.

Proposition 4 states that adding a firm to the market will guarantee a deterioration in credibility, but only for a limited amount of time. In fact, the addition of a firm may result in an improvement in credibility during later periods. This phenomenon is captured by Figure 5. This figure plots simulations of α_n under $N = 5$ and $N = 6$, respectively, holding all other parameters fixed. While the addition of a firm lowers credibility in early periods, it improves credibility in later periods.

To understand this result, note that an additional firm will effect two separate changes to the market. First, each firm faces greater competition, and thus a greater risk of being preempted. This change is precisely what was captured in our earlier exercise regarding the effects of competition. As illustrated by Figure 3, this change will cause a deterioration in credibility. However, an additional firm also increases the market's total ability to learn. This change is captured by our comparative static on λ, which shows that an increase in learning ability will cause an improvement in credibility. Thus, the effect of an additional firm can be understood as the combination of two countervailing forces: higher competition and a higher ability to learn within the market.

To understand why the credibility-diminishing effect of higher competition must dominate in early periods, we must compare the relative magnitudes of these the two countervailing forces. Figure 4 illustrates that while credibility is pointwise higher at every $t > 0$ under high λ, this difference is negligible in early periods. This is due to the fact that firms learn gradually over time, and thus it takes time for differences in learning ability to substantially impact firms' beliefs. Meanwhile, as illustrated by Figure 3, an increase in competition will have a non-negligible impact on credibility even when $t = 0$. For this reason, the impact of higher competition must dominate in early periods, resulting in a net reduction in credibility. However, as time passes and the effect of faster learning grows, a reversal may take place, i.e., there may be a net improvement in credibility. Such a scenario is precisely what is depicted by Figure 5.

6. Extension: Heterogeneous Ability

In this section, I consider an extension in which firms are heterogeneous in their abilities to learn. Doing so will shed light on how a firm's credibility correlates with it's ability in equilibrium.

Formally, this extended model is identical to the model above except for three changes. First, rather than assuming that each firm is endowed with the same ability λ, I assume that each firm i is endowed with an individual-specific ability λ^i. As with all other parameters, I assume that these individual-specific abilities are common knowledge. Second, to simplify our analysis for this exercise, I will restrict attention to a winner-takes-all setting: i.e., I assume that $k_n = 0$ for all $n > 1$. Finally, I relax our assumption that the equilibrium is symmetric. Thus, different firms (and in particular, firms with different abilities) may play different strategies in equilibrium and are thus a firm's credibility is individual-specific. Accordingly, I let α^i denote the credibility of firm i.

I obtain an intuitive result: a firm's ability correlates positively with its credibility in equilibrium. This is stated formally as Proposition 5.

Proposition 5. *For all (i, j) such that $\lambda^i < \lambda^j$, $\alpha_1^i(p(t)) \leq \alpha_1^j(p(t))$. Furthermore, this inequality is strict whenever $\alpha_1^i(p(t)) < 1$.*

Proposition 5 states that regardless of when a report is made, a firm with higher ability will be more credible.[12] Furthermore, a high ability firm will be strictly more credible than a low ability firm whenever firms are not fully truthful.

Let us now consider why this correlation arises. First, note that high ability firms are able to confirm a story more quickly and thus, all else equal, pose a greater preemptive

[12] This claim restricts attention to the first firm to report, because by the winner-takes-all assumption, all following senders will never fake, i.e., $\alpha_n^i(p) = 1$ whenever $n > 1$.

threat in equilibrium. This in turn implies that in comparison to a high-ability firm, a low-ability firm faces a greater preemptive threat. Thus, the low-ability firm finds immediate faking more advantageous. In light of this, the firms' credibilities must adjust in such a way to preserve their respective indifference conditions. This is achieved endogenously by means of a lower credibility for the low-ability firm, which ensures that it has less to gain from faking immediately.

7. Conclusion

In this paper, I presented a dynamic model of breaking news to understand the nature of reporting errors. I sought to explain how strategic forces that could induce firms to err. In this setting, errors were driven by two qualities of the breaking news environment: a firm's lack of commitment power as well as competition. I find that competition induces firms to err through two separate channels: preemptive motives and observational learning. While preemptive motives can give rise to errors by encouraging firms to report hastily, observational learning can cause an existing error to propagate through the market.

The second key objective was to understand the dynamics of reporting errors. In equilibrium, these dynamics take two forms. First, firms become gradually more truthful over time as long as no new reports are made. Furthermore, a firm's credibility gradually increases whenever preemptive motives are at play. Importantly, this improvement in credibility incentivizes firms to take their time, and thus counteracts the haste-inducing effects of preemption. Dynamics also take the form of discrete changes in the firm's behavior and credibility which are triggered by a rival report. In particular, I document a copycat effect, where a report by one firm can induce a surge in faking by other firms in the market.

While I consider breaking news specifically, this model provides broader insight into how preemptive concerns can affect the quality of information provided by experts. To understand how preemption impacts information provision more broadly is a topic that warrants further investigation.

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Appendix A Discrete-time approximation of α

In this section, we formally justify equation (2), our equilibrium formula for $\alpha_n(p)$, by showing that it is the limit of Bayes-consistent beliefs under a discretized version of the game presented in section (2). For any $\varepsilon > 0$, let the ε-approximation of the game be identical to the game presented in section (2), except with the following modification: any report made by a firm on $[0, \varepsilon]$ is observed by all other players (including the consumer) at ε. Formally, rather than observing t_i, the players observe $\tilde{t}_i$, where

$$\tilde{t}_i = max\{t_i, \varepsilon\}$$

At any (p, n) that is on-path, let $\alpha_n^\varepsilon(p)$ denote the firm's credibility, i.e., the consumer's belief that $s_i \leq \varepsilon$ given that $\tilde{t}_i = \varepsilon$, under the ε approximation of the game. Let us define α_n^ε to be the right-limit of the α_ε, formally:

$$\alpha_n(p) \equiv \lim_{\varepsilon \to 0+} \alpha_n^\varepsilon(p)$$

We now establish that on-path, $\alpha_n(p)$ is given by (2).

Claim 2. *For any (p, n) on-path,*

$$\alpha_n(p) = \begin{cases} \frac{\lambda p}{\lambda p + F'_{p,n}(0+)} & \textit{if } F_{p,n}(0) = 0 \\ 0 & \textit{if } F_{p,n}(0) > 0 \end{cases}$$

Proof. For any $\varepsilon > 0$, $\alpha_n^\varepsilon(p, n)$ is uniquely determiend by Bayes Rule and given by

$$\alpha_\varepsilon(p, n) = \frac{p(1 - e^{-\lambda\varepsilon})}{p(1 - e^{-\lambda\varepsilon}) + F_{p,n}(\varepsilon)e^{-\lambda\varepsilon}}.$$

First, consider the case where $F_{p,n}(0) = 0$. In this case, it follows from L'Hôpital's Rule that:

$$\lim_{\varepsilon \to 0+} \alpha_\varepsilon(p, n) = \frac{\lambda p}{\lambda p + F'_{p,n}(0+)}$$

Next, consider the case where $F_{p,n}(0) > 0$. In this case, we obtain

$$\lim_{\varepsilon \to 0+} \alpha_\varepsilon(p, n) = \frac{0}{0 + \lim_{\varepsilon \to 0+} F_{p,n}(\varepsilon)} = 0$$

where the final equality follows from the fact that $\lim_{\varepsilon \to 0+} F_{p,n}(\varepsilon) = F_{p,n}(0) > 0$.

□

Appendix B Beliefs in equilibrium

We begin by stating some relevant properties and notation regarding the players' beliefs about the state.

First, we remark that at all times t and histories H, all players, with the exception of those who have already reported, must hold a common belief about the state. We omit a formal proof as this follows directly from our selection assumption that $F_{1,n}(0) = 1$. This assumption implies that it is common knowledge that all firms who have not yet reported have not observed a conclusive signal. Thus, all such players, in addition to the consumer, share the same information set, and thus a common belief about the state.

Next, fixing an initial common belief p, and number of remaining firms n, we define two conditional beliefs, $p(s)$ and $p^i(s)$, which we will reference frequently in the analysis that follows. We let $p(s)$ denote the common updated belief, conditional on no new reports being made after s time passes. It follows from Bayes Rule that

$$p(s) = \frac{pe^{-n\lambda s}}{pe^{-n\lambda s} + (1-p)} \tag{4}$$

Meanwhile, we let $p^i(s)$ denote the common updated belief, conditional on the event that player i report at s, and no other reports were made. Again, $p^i(s)$ follows directly from Bayes Rule, given α:

$$p^i(s) = \alpha_n(p(s)) + (1 - \alpha_n(p(s))p(s) \tag{5}$$

To understand how $p^i(s)$ is computed, note that if a report is made after time s has passed, conditioning on the event that i's report was informed, the common belief will update to 1. However, conditioning on the event that i was uninformed when making the report, the report would have no impact on the common belief, which would thus be given by $p(s)$. Thus, $p^i(s)$ is given by the weighted sum of these two beliefs, where the weighting is specified by the belief that the report was informed, i.e., $\alpha_n(p(s))$.

Appendix C The firm's problem

Before proceeding, we define a useful object, the *first report distribution* Ψ. Formally, fixing a (p, n), $\Psi^i(s)$ denotes the probability that player i reported at or before s and was not preceded by any of the remaining firms in doing so. Fixing a strategy profile $(F^1_{p,n}, ..., F^n_{p,n})$, it is given by:

$$\Psi^i(s) = p\int_0^s e^{-\lambda r(N-n)} \prod_{j\neq i}(1-F^j_{p,n}(r))d(e^{-\lambda r}(F^i_{p,n}(r)-1)) + (1-p)\int_0^s \prod_{j\neq i}(1-F^j_{p,n}(r))dF^i_{p,n}(r)$$

The first integral of the expression denotes the probability that i is the first firm conditional on $\theta = 1$, while the second integral denotes the same probability conditional on $\theta = 0$.

$\Psi^i(s)$ is then the weighted sum of these two probabilities, where the weight is given by the common belief p about θ. Note that while Ψ is a function of the strategy profile, p, and n, we omit this dependence for brevity.

The firm's problem is defined recursively as follows. Fix a firm i, n, p, α, and continuation value function $V_{\cdot,n+1}$. Trivially, $V_{p,0} = 0$ for all p. Assume all firms $j \neq i$ play the same strategy F, and let $-i$ to generically to refer to $j \neq i$. Then i's expected payoff from playing strategy F^i at (p, n) is given by:

$$V_{p,n}(F^i) = \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s) + (N - n) \int_0^\infty V_{p^{-i}(s),n+1} d\Psi^{-i}(s)$$

Note that first integral of this expression is firm i's expected payoff from reporting, when it is the first to do so. Meanwhile, the second integral is the firm's expected payoff in the circumstance where it is preempted. The firm's problem at (p, n) is then given by the following:

$$\max_{F^i \in \mathcal{F}} V_{p,n}(F^i),$$

where $\mathcal{F}$ denotes the set of permissible distributions, i.e., those that are piecewise continuously differenetiable, right-differentiable, and that satisfy the selection criterion. We further define $V_{p,n} = \max_{F^i \in \mathcal{F}} V_{p,n}(F^i)$.

Appendix D Regularity of F and α

Proof of Lemma 1. We will begin by showing that at all (p, n) on-path such that $p < 1$, $F_{p,n}$ is continuous at 0. To this end, suppose by contradiction that $F_{p,n}$ exhibits discontinuous at 0. By the right-continuity of $F_{p,n}$, $F_{p,n}(0) > 0$. Because (p, n) is on path, by (2), $\alpha_n(p) = 0$. Furthermore, it follows by (5) that $p^i(0) = p$. Recalling that we are restricting attention to symmetric equilibria, let Ψ denote the first-report distribution at (p, n) under the equilibrium strategy profile in which all firms who have not yet reported play $F_{p,n}$. Because $F_{p,n}(0) > 0$, $\Psi^j(0) > 0$ for all j who have not yet reported.

Now define the following deviation $\hat{F}_{p,n}$. This strategy is identical to $F_{p,n}$, except that all the mass that $F_{p,n}$ places on 0 is shifted to ∞:

$$\hat{F}_{p,n}(s) = \begin{cases} F_{p,n}(s) - F_{p,n}(0) & \text{if } s < \infty \\ 1 & \text{if } s = \infty \end{cases}$$

Now, fix some i who has not yet reported. Let $\hat{\Psi}$ denote the first-report distribution at (p, n) under the strategy profile where i plays $\hat{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$. By definition, for all

$s \geq 0$,

$$\hat{\Psi}_i(s) = \Psi^i(s) - \Psi^i(0).$$

Thus

$$\int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\hat{\Psi}_i(s) = \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s) + \beta(1-p^i(0))\Psi^i(0)$$
$$> \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s).$$

Again by definition, for all $s \geq 0$,

$$\hat{\Psi}_{-i}(s) = \Psi^{-i}(s) + X(s),$$

where

$$X(s) \equiv \Psi^i(0)[p\int_0^s (1-F_{p,n})^{n-2}(1-\hat{F}_{p,n}(r))e^{-\lambda r}d(e^{-\lambda r}(F_{p,n}(r)-1))$$
$$+(1-p)\int_0^s (1-F_{p,n}(r))^{n-2}(1-\hat{F}_{p,n}(r))dF_{p,n}(r)]$$

Since $X(s)$ is weakly increasing in s,

$$\int_0^\infty V_{p^{-i}(s),n+1}d\hat{\Psi}_{-i}(s) - \int_0^\infty V_{p^{-i}(s),N=1}d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s),n+1}dX(s) \geq 0.$$

where the final inequality follows from the fact that $X(s)$ is increasing in s and $V_{p^{-i}(s),n+1} \geq V_{p^{-i}(s),n+1}(delta_\infty) \geq 0$.

Combining the above two inequalities we have

$$V_{p,n}(\hat{F}_{p,n}) = \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\hat{\Psi}_i(s) + (N-n)\int_0^\infty V_{p^{-i}(s),n+1}d\hat{\Psi}_{-i}(s)$$
$$> \int_0^\infty [k_n\alpha_n(p(s)) - \beta(1-p^i(s))]d\Psi^i(s) + (N-n)\int_0^\infty V_{p^{-i}(s),n+1}d\Psi^{-i}(s) = V_{p,n}(F_{p,n})$$

Thus, i can profitably deviate at (p,n): contradiction.

We will now show that for all (p,n) on-path such that $p<1$, $F_{p,n}$ must be continuous at all t. Suppose by contradiction that it is not. Let t denote the time at which a discontinuity occurs. Because $F_{p,n}$ is increasing and right-differentiable by assumption, this must be a jump discontinuity, i.e.,

$$\lim_{r\to t-} F_{p,n}(r) < F_{p,n}(t)$$

By (1),

$$F_{p(t),n}(0) = \frac{F_{p,n}(t) - \lim_{r \uparrow t} F_{p,n}(r)}{1 - \lim_{r \uparrow t} F_{p,n}(r)} > 0.$$

But then, this implies that $F_{p(t),n}$ is discontinuous at 0, contradicting the above.

Part (b) of the statement follows directly from (2). □

Lemma 2. *For any (p,n) on-path,*

- $\alpha_n(p) \geq \overline{\alpha}_n(p) \equiv \min\{\beta(1-p)/k_n, 1\}$
- $F'_{p,n}(0+) \leq \overline{f} \equiv \lambda p(\frac{1}{\overline{\alpha}(p,n)} - 1)$

Proof of Lemma 2. We begin by showing the first point above. The second point follows by definition of $\alpha_n(p)$.

First, suppose by contradiction that there exists a (p,n) on-path such that

$$\alpha_n(p) < \min\{\beta(1-p)/k_n, 1\}$$

Recalling that $p(s)$ is given by (4), we begin by claiming that for all s sufficiently small, $(p(s), n)$ is on-path. Suppose not by contradiction. Since (p,n) is on-path by assumption, this implies that $F_{p,n}(0) = 1$, which contradicts Lemma 1. It thus follows from (2), combined with the piecewise twice differentiability and right-differentiability of $F_{p,n}$, that $\alpha(p(s), n)$ is continuous in some right-neighborhood of $s = 0$. Formally, there exists an $\varepsilon > 0$ such that for all $s \in [0, \varepsilon]$,

$$k_n \alpha_n(p(s)) < \beta(1-p).$$

Next, I claim that $F_{p,n}(\varepsilon) > 0$. Suppose this is not true by contradiction. Then, it follows that $F_{p,n}(s) = 0$ for all $s \in [0, \varepsilon]$, implying by definition of α that $\alpha_n(p) = 1$, contradicting our assumption that $\alpha_n(p) < 1$.

Now, define the following deviation $\tilde{F}_{p,n}$, which shifts the mass $F_{p,n}$ places on $[0, \varepsilon]$ to ∞:

$$\tilde{F}_{p,n}(s) = \begin{cases} 0 & \text{if } s \in [0, \varepsilon] \\ F_{p,n}(s) - F_{p,n}(\varepsilon) & \text{if } s \in (\varepsilon, \infty) \\ 1 & \text{if } s = \infty \end{cases}$$

The admissibility of $\tilde{F}_{p,n}$ follows from the admissibility of $F_{p,n}$. We now wish to show that $\tilde{F}_{p,n}$ is a profitable deviation at (p,n). Let Ψ denote the first-report distribution under the strategy profile where all players play $F_{p,n}$, and let $\tilde{\Psi}$ denote the first-report distribution under the strategy profile where i plays $\tilde{F}_{p,n}$ and all $j \neq i$ play $F_{p,n}$.

By definition of Ψ,

$$\tilde{\Psi}_i(s) = \Psi^i(s) - X(s)$$

where

$$X(s) = \begin{cases} p \int_0^s e^{-\lambda r(N-n)}(1 - F_{p,n}(r))^{N-n} d(e^{-\lambda r}(F_{p,n}(r) - 1)) + (1-p) \int_0^s (1 - F_{p,n}(r))^{N-n} dF_{p,n}(r) & \text{if } s \in [0, \varepsilon] \\ X(\varepsilon) & \text{if } s > \varepsilon \end{cases}$$

Now, note that $X(s)$ is weakly increasing in s. Note further that because $F_{p,n}(\varepsilon) \in (0,1]$, it follows that $F_{p,n}(s)$ strictly increases on $[0, \varepsilon]$. Thus, $X(s)$ is strictly increasing at some $s \in [0, \varepsilon]$. Now, by the above definition:

$$\int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\tilde{\Psi}_i(s) - \int_0^\infty [k_n \alpha_n(p(s)) - \beta(1 - p^i(s))] d\Psi^i(s)$$
$$= \int_0^\varepsilon [k_n \alpha_n(p(s)) - \beta(1 - p(s))] dX(s) > 0$$

where the strict inequality follows from the fact that $X(s)$ is strictly increasing on $[0, \varepsilon]$ and the above-established fact that $k_n \alpha_n(p(s)) < \beta(1 - p(s))$ for all $s \in [0, \varepsilon]$.

Next, let us examine $\tilde{\Psi}(-i, s)$. It again follows from the definition of Ψ that

$$\tilde{\Psi}_{-i}(s) = \Psi^{-i}(s) = Y(s)$$

where

$$Y(s) = p \int_0^s [e^{-\lambda r}(1 - F_{p,n}(r))]^{n-2} F(\min\{r, \varepsilon\}) d(e^{-\lambda r}(F_{p,n}(r) - 1)) +$$
$$(1-p) \int_0^s (1 - F_{p,n}(r))^{n-2} F_{p,n}(\min\{r, \varepsilon\}) dF_{p,n}(r)$$

Thus,

$$\int_0^\infty V_{p^{-i}(s), n+1} d\tilde{\Psi}{-i}(s) - \int_0^\infty V_{p^{-i}(s), n+1} d\Psi^{-i}(s) = \int_0^\infty V_{p^{-i}(s), n+1} dY(s) \geq 0$$

where the final inequality follows for from the fact that $Y(s)$ is increasing in s and $V_{p^{-i}(s), n+1} \geq 0$. Combining the previous two inequalities, we obtain that

$$V_{p,n}(\tilde{F}_{p,n}) > V_{p,n}(F_{p,n})$$

and thus i can profitably deviate at (p, n). Contradiction. $\square$

Lemma 3. *If $\alpha_n(p) < 1$ and (p, n) is on-path, then there exists an $\varepsilon > 0$ such that*

$$V_{p,n} = V_{p,n}(\delta_s) \textit{ for all } s \in [0, \varepsilon] \cup \infty.$$

Proof of Lemma 3. Assume that $\alpha_n(p) < 1$. Note that by the right twice-differentiability of $F_{p,n}$, and by (2), that $\alpha_n(p(s))$ is right-continuous in s. Thus, there exists an $\varepsilon > 0$ and $d > 0$ such that

$$\alpha_n(p(s)) < 1 - d \text{ for all } s \in [0, \varepsilon].$$

I claim that for all $s \in [0, \varepsilon)$, $V_{p,n} = V_{p,n}(\delta_s)$. Suppose to the contrary that for some $s \in [0, \varepsilon)$,

$$V_{p,n}(\delta_s) < V_{p,n}$$

Now, I claim that $V_{p,n}(\delta_s)$ is right-continuous in s. To see why this is the case, note that by definition,

$$V_{p,n}(\delta_s) = \int_0^s k_n \alpha_n(p(r)) d\Psi^i(r) + (N - n) \int_0^s V_{p^i(r),n} d\Psi^{-i}(r) + \\ (1 - \sum_j \Psi^j(s))[k_n \alpha(p(s), n) - \beta(1 - p(s))]$$

Where $\Psi^j(s)$ is the first-report distribution that arises when i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. The right-continuity with respect to s then follows from the absolute continuity of Ψ^j (which follows from Lemma 1), as well as the right-continuity of $\alpha(p(s), n)$ with respect to s, which follows from the right-continuity of $F_{p,n}(s)$, which follows by assumption.

Given this right-continuity, some $\varepsilon' \in (0, \varepsilon - s)$ and $x > 0$ such that

$$V_{p,n} - V_{p,n}(\delta_r) > x \text{ for all } r \in [s, s + \varepsilon']$$

Note that there must exist some $s^* \in [0, \infty]$ such that $V_{p,n} = V_{p,n}(\delta_{s^*})$. Then, define the following deviation $\tilde{F}_{p,n}$ which shifts all the mass from $[s, s + \varepsilon']$ to s^*. Specifically, when $s^* < s$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(t) + F_{p,n}(s + \varepsilon) - F_{p,n}(s) & \text{if } t \in [s^*, s] \\ F_{p,n}(s + \varepsilon) & \text{if } t \in (s, s + \varepsilon'] \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Meanwhile, when $s^* > s+\varepsilon$:

$$\tilde{F}_{p,n}(t) = \begin{cases} F_{p,n}(s) & \text{if } t \in [s, s+\varepsilon] \\ F_{p,n}(t) - [F_{p,n}(s+\varepsilon') - F_{p,n}(s)] & \text{if } t \in (s+\varepsilon', s^*) \\ F_{p,n}(t) & \text{otherwise.} \end{cases}$$

Now, by definition:

$$V_{p,n}(\tilde{F}_{p,n}) = V_{p,n}(F_{p,n}) + \int_s^{s+\varepsilon'} V_{p,n}(\delta_{s^*}) - V_{p,n}(\delta_r) dF_{p,n}(r) \geq V_{p,n}(F_{p,n}) + x\varepsilon' > V_{p,n}(F_{p,n})$$

Thus, $\tilde{F}_{p,n}$ serves as a profitable deviation. Contradiction.

It remains to show that $V_{p,n} = V_{p,n}(\delta_\infty)$. Suppose by contradiction that $V_{p,n} > V_{p,n}(\delta_\infty)$. It follows that $\lim_{t\to\infty} F_{p,n}(t) = 0$, because otherwise, the firm could profitably deviate by placing no mass on $t = \infty$. But this implies that for some $s \in (0, \infty]$,

$$\lim_{t\to s-} F'_{p,n}(t+) = \infty \Rightarrow \lim_{t\to s-} \alpha_n(p(t)) = 0,$$

which contradicts Lemma 2. □

Lemma 4. *$\alpha_n(p(s))$ is continuous in s for all (p, n) on path such that $s > 0$.*

Proof of Lemma 4. Fix a (p, n) on-path. I first claim that for all $s \geq 0$,

$$\alpha_n(p(s)) = \frac{\lambda p}{\lambda p + \frac{F'_{p,n}(s+)}{1-F_{p,n}(s)}} \tag{6}$$

To see why this must hold, note that it follows from Lemma 2 that $(p(s), n)$ is on-path for all $s \geq 0$. Thus, by Lemma 1, $F_{p(s),n}(0) = 0$, and thus it follows from (2) that for all $s \geq 0$,

$$\alpha_n(p(s)) = \frac{\lambda p(s)}{\lambda p(s) + F'_{p(s),n}(0+)}.$$

Next, it follows from (1) that

$$F'_{p(s),n}(0+) = \frac{F'_{p,n}(s+)}{1 - F_{p,n}(s)}.$$

Combining the previous two equations yields (6). It thus follows from the right-continuity of $F_{p,n}$ that $\alpha_n(p(s))$ is right-continuous in s. It remains to show that it is left-continuous.

Suppose by contradiction there exists an s such that $\alpha_n(p(s))$ is left-discontinuous. Then there exists some $d > 0$ such that for all $\varepsilon > 0$, there exists an $s_\varepsilon \in (s - \varepsilon, s)$ such that

$$|\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s))| > d.$$

First consider the case where for all $\varepsilon > 0$, $\alpha_n(p(s_\varepsilon)) - \alpha_n(p(s)) > d$. I begin by claiming that for all $\varepsilon > 0$,

$$V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}). \tag{7}$$

To this end, first note that there exists some $s^* \in (s, \infty]$ such that $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_{s^*,s_\infty})$. To see why this must hold, suppose not, by contradiction. Then it must be that $F_{p(s_\varepsilon),n}$ places full support on $[s_\varepsilon, s]$, and thus, either Lemma 1 or (2) would be violated. Thus, we have

$$\begin{aligned} V_{p(s_\varepsilon),n} = \int_0^{s-s_\varepsilon} k_n\alpha(p(r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(r),n+1}d\Psi^{-i}(r) + \\ (1 - \sum_j \Psi^j(s))V_{p(s),n}(\delta_{s^*-s}) \le \int_0^{s-s_\varepsilon} k_n\alpha(p(r))d\Psi^i(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(r),n+1}d\Psi^{-i}(r) \\ + (1 - \sum_j \Psi^j(s))V_{p(s),n}(\delta_0) = V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) \end{aligned}$$

where Ψ is the first-report distribution associated witht he strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p(s_\varepsilon),n}$. Note that the inequality follows from the fact that $\alpha_n(p(s)) < 1$, and thus by Lemma 3, $V_{p(s),n} = V_{p(s),n}(\delta_0)$. However, note that for all $\varepsilon > 0$,

$$\begin{aligned} V_{p(s_\varepsilon),n}(\delta_{s-\varepsilon}) = \int_0^{s-s_\varepsilon} k_n\alpha_n(p(r))d\Psi^I(r) + (N-n)\int_0^{s-s_\varepsilon} V_{p^i(r),n+1}d\Psi^{-i}(r) \\ + (1 - \sum_j \Psi^j(r))[k_n\alpha_n(p(s), n) - \beta(1-p(s))] \end{aligned}$$

Because the Ψ^j are absolutely continuous,

$$\lim_{\varepsilon \to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = k_n\alpha_n(p(s), n) - \beta(1-p(s))$$

Thus, for all $\varepsilon > 0$ sufficiently small, $V_{p(s_\varepsilon),n}(\delta_0) > V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon})$, contradicting (7).

Next, consider the case where for all $\varepsilon > 0$, $\alpha_n(p(s)) - \alpha_n(p(s_\varepsilon)) > d$. As noted above, $\lim_{\varepsilon \to 0} V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) = V_{p(s),n}(\delta_0)$. Thus, for ε sufficiently small,

$$V_{p(s_\varepsilon),n}(\delta_{s-s_\varepsilon}) > k_n\alpha_n(p(s_\varepsilon)) - \beta(1-p(s_\varepsilon)) = V_{p(s_\varepsilon),n}(\delta_0)$$

However, since for all $\varepsilon > 0$, $\alpha_n(p(s_\varepsilon)) < 1$. By Lemma 3, $V_{p(s_\varepsilon),n} = V_{p(s_\varepsilon),n}(\delta_0)$. Contradiction. □

Appendix E Characterization proofs

Proof of Proposition 1. We begin by showing that $\alpha_n(p) = 1$ whenever $k_n < \beta$ and $p \leq p_n^* \equiv \frac{k_n - \beta}{k_n/n - \beta}$.

To this end, fix any n, and suppose that $k_n < \beta$. We begin by showing that for all $q < \frac{\beta - k_n}{\beta}$, $\alpha_n(q) = 1$. Note that for all such q

$$V_{q,n}(\delta_0) = k_n\alpha_n(q) - \beta(1-q) \leq k_n - \beta(1-q) < k_n - \beta(1 - \frac{\beta - k_n}{\beta}) = 0.$$

Since $V_{q,n} \geq V_{q,n}(\delta_\infty) \geq 0$, it follows $V_{q,n} > V_{q,n}(\delta_0)$. Thus, by Lemma 3, $\alpha_n(q) = 1$. Now, let

$$q_n^* \equiv \sup\{p | \alpha_n(q) = 1 \text{ for all } q < p\}$$

It follows from the above that $q_n^* \geq \frac{\beta - k_n}{\beta} > 0$. Now suppose by contradiction that $q_n^* < p_n^*$. By Lemma 4, there exists an $\varepsilon > 0$ such that for all $p \in (q_n^*, q_n^* + \varepsilon)$, $\alpha_n(p) < 1$, and thus, by Lemma 3

$$V_{p,n} = V_{p,n}(\delta_0) = k_n\alpha_n(p) - \beta(1-p)$$

Thus, it follows from Lemma 4 that

$$\lim_{p \to q_n^*+} V_{p,n} = k_n - \beta(1 - q_n^*) \tag{8}$$

By definition of V, because by Lemma 1 $F_{p,n}$ is absolutely continuous, it follows that $V_{p,n}(\delta_\infty)$ is as well, and thus:

$$\lim_{p \to q_n^*+} V_{p,n}(\delta_\infty) = V_{q_n^*,n}(\delta_\infty) = \frac{k_n q_n^*}{n} \tag{9}$$

In for δ_∞ to not serve as a profitable deviation for $p \in (q_n^*, q_n^* + \varepsilon)$, it must be that for all such p, $V_{p,n}(\delta_0) \geq V_{p,n}(\delta_\infty)$. Taking a limit we obtain that

$$\lim_{p \to q_n^*+} V_{p,n} \geq \lim_{p \to q_n^*+} V_{p,n}$$

Substituting (8) and (9) above, we obtain that $\frac{k_n q_n^*}{n} \leq k_n - \beta(1 - q_n^*)$. However, $k_n \leq \beta$ and $q_n^* < p$ implies that $\frac{k_n q^*}{n} > k_n - \beta(1-q)$. Contradiction.

Next, we show that $\alpha_n(p) < 1$ whenever $\beta \leq k_n$ or $p > p_n^*$. To this end, assume $\beta \leq k_n$

or $p > p_n^*$. Assume by contradiction that $\alpha_n(p) = 1$. Also assume by induction that if $n > 1$, then the statement holds for $n + 1$.

First, consider the case where $\alpha_n(q) = 1$ for all $q < p$. By (2), this implies that $F'(q, n) = 0$. Furthermore, by Lemma 1, this implies that $F_{p,n}(s) = 0$ for all $s > 0$, i.e., $F_{p,n} = \delta_\infty$. However,

$$V_{p,n}(\delta_0) = k_n - \beta(1-p) > \frac{k_n p}{n} = V_{p,n}(\delta_\infty),$$

where the above strict inequality follows from the above assumption that either $\beta \leq k_n$ or $p > p_n^*$.

Next, consider the case where $\alpha_n(q) < 1$ for some $q < p$. By Lemma 4, for all $\varepsilon > 0$ sufficiently small, there exists some $\overline{p} < p$ and $\overline{s} > 0$ such that $\alpha_n(\overline{p}) \in (1-\varepsilon, 1)$ and $\alpha_n(q)$ is strictly increasing on $[\overline{p}(\overline{s}), \overline{p}]$. By Lemma 3, there exists some $\Delta \in (0, s)$ such that

$$V_{\overline{p},n}(\delta_\Delta) = V(\overline{p}, n, \delta_0).$$

By definition,

$$V(\overline{p}, n, \delta_\Delta) = \int_0^\Delta k_n \alpha_n(\overline{p}(s)) d\Psi^i(s) + (N-n) \int_0^\Delta V(\overline{p}^i(s), n+1) d\Psi^{-i}(s) +$$
$$(1 - \sum_j \Psi^j(\Delta))(k_n \alpha_n(\overline{p}(\Delta)) - \beta(1 - \overline{p}(\Delta))$$

where Ψ is the first-report distribution associated with the strategy profile where i plays δ_Δ and all $j = i$ play $F_{p,n}$. Meanwhile,

$$\begin{aligned}V(\overline{p}, n, \delta_0) &= k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}) \\ &= \int_0^\Delta k_n \alpha_n(\overline{p}) d\Psi^i(s) + (N-n) \int_0^\Delta k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}^i(s)) d\Psi^{-i}(s) \\ &+ (1 - \sum_j \Psi^j(\Delta))(k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}(\Delta))\end{aligned}$$

Thus, in order to preserve the above equality, for some $r \in (0, \overline{s})$,

$$k_n \alpha_n(\overline{p}) - \beta(1 - p^i(r)) < V(\overline{p}^i(r), n+1). \tag{10}$$

First, consider the case where $\alpha_{n+1}(\overline{p}^i(r))) < 1$. Then, for $\varepsilon > 0$ sufficiently small

$$V_{\overline{p}^i(r),n+1} = V_{\overline{p}^i(r),n+1}(\delta_0) = k_{n+1}\alpha_{n+1}(\overline{p}^i(r)) - \beta(1 - \overline{p}^i(r)) < k_n \alpha_n(\overline{p}) - \beta(1 - \overline{p}^i(r))$$

where the first equality follows from Lemma 3. Thus, equation (10) is violated. Contradiction.

Next, consider the case where $\alpha_{n+1}(\overline{p}^i(r)) = 1$ and $\beta < k_n$. By the inductive assumption, it follows that $\alpha_{n+1}(q) = 1$ for all $q \leq \overline{p}^i(s)$. Thus, $F_{\overline{p}^i(s),n+1} = \delta_\infty$. So, we have that for ε sufficiently small:

$$V_{\overline{p}^i(r),n+1} = V_{\overline{p}^i(r),n+1}(\delta_\infty) = \frac{k_{n+1}\overline{p}^i(r)}{N-n} \leq \overline{p}^i(r)k_n\alpha_n(\overline{p}) + (1-\overline{p}^i(s))k_n\alpha_n(\overline{p}) - \beta)$$
$$= k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(s))$$

Again, this is a contradiction of (10).

Finally, consider the case where $\alpha_{n+1}(\overline{p}^i(r)) = 1$ and $\beta \geq k_n$. Recall by Proposition 1 that $\alpha_n(q) = 1$ for all $q \geq p_n^*$. Thus, because $\alpha_n(\overline{p}) < 1$, it follows from (4) that $\alpha_n(\overline{p}(s))$ must be strictly increasing in s for some $s > r$. Formally, let

$$r' = \inf\{s > r | \alpha_n(\overline{p}(s)) \text{ is strictly increasing}\}.$$

First, we claim that

$$k_n\alpha_n(\overline{p}(r')) - \beta(1-\overline{p}^i(r)) < V_{\overline{p}^i(r),n+1} \tag{11}$$

By the inductive assumption, since $\alpha_{n+1}(\overline{p}^i) = 1$, it must be that $\alpha_{n+1}(q) = 1$ for all $q < \overline{p}^i(r)$. Because $\alpha_n(\overline{p}(s))$ is weakly decreasing for $s \in [r, r']$, it follows by definition of $\overline{p}^i(s)$ that $\overline{p}^i(s) < \overline{p}^i(r)$ for all $s \in [r, r']$. Thus, for all $s \in [r, r']$

$$V_{\overline{p}^i(s),n+1} = \frac{k_{n+1}\overline{p}^i(s)}{N-n}.$$

It follows from this that for all $s \geq r$,

$$k_n\alpha_n(\overline{p}(s)) - \beta(1-\overline{p}^i(s)) < V_{\overline{p}^i(s),n+1}$$
$$\Leftrightarrow k_n\alpha_n(\overline{p}(s)) - \beta(1-\overline{p}^i(s)) < \frac{k_{n+1}\overline{p}^i(s)}{N-n}$$
$$\Leftrightarrow \overline{p}^i(s) < \frac{\beta - k_n\alpha_n(\overline{p}(r))}{\beta - k_{n+1}/(N-n)}$$

Now, because $\alpha_n(\overline{p}(s))$ is strictly decreasing on $s \in [0, r]$.

$$k_n\alpha_n(\overline{p}(r)) - \beta(1-p^i(r)) < k_n\alpha_n(\overline{p}) - \beta(1-\overline{p}^i(r)) < V_{\overline{p}^i(r),n+1}$$

where the second inequality was established above. Thus we have

$$\overline{p}^i(r') < \overline{p}^i(r) < \frac{\beta - k_n\alpha_{n+1}(\overline{p}(r))}{\beta - k_{n+1}/(N-n)} < \frac{\beta - k_n\alpha_{n+1}(\overline{p}(r'))}{\beta - k_{n+1}/(N-n)}$$

which implies (11).

It follows from this that there exists an $r'' > r'$ such that for all $s \in [r', r'']$, $\alpha_n(\overline{p}(s))$ is weakly decreasing and $V_{\overline{p}^i(s),n+1} > k_n\alpha_n(\overline{p}(r')) - \beta(1 - p^i(s))$. Now I claim that

$$V_{\overline{p}(r'),n}(\delta_0) < V_{\overline{p}(r'),n}(\delta_{r''-r'}).$$

To see why this must be true, note that by definition,

$$\begin{aligned} V_{\overline{p}(r'),n}(\delta_{r''-r'}) - V_{\overline{p}(r'),n}(\delta_0) = \int_{r'}^{r''} k_n[\alpha_n(p(s)) - \alpha_n(p(r'))]d\Psi(i,s) + \\ \int_{r'}^{r''} [V_{p^i(s),n+1} - (k_n\alpha_n(p(r')) - \beta(1 - p^i(s)))]d\Psi^{-i}(s) \\ + \sum_j (\Psi(j, r'') - \Psi(j, r'))k_n(\alpha_n(p(r'')) - k_n\alpha_n(p(r'))) \end{aligned}$$

Since $\alpha_n(p(s)) \geq \alpha_n(p(r'))$ and $V_{p^i(s),n+1} > k_n\alpha_n(p(r')) - \beta(1 - p^i(s))$ $s \in [r', r'']$, it follows that $V_{\overline{p}(r'),n}(\delta_{r''-r'}) - V_{\overline{p}(r'),n}(\delta_0) > 0$. However, this contradicts Lemma 3. □

Proof of Proposition 2. Proof by induction. Fix an n, and assume that $\alpha_m(p)$ satisfies the above for all $m < n$ such that (p, m) is on-path.

We begin by showing that (ODE) must hold whenever $\alpha_n(p) < 1$. To this end, assume that $\alpha_n(p) < 1$. Then, by Lemma 3, there exists an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,n}(\delta_\Delta) - V_{p,n}(\delta_0)}{\Delta} = 0 \tag{12}$$

Recall that by definition of V, that

$$V_{p,n}(\delta_0) = k_n\alpha_n(p) - \beta(1 - p).$$

Meanwhile

$$\begin{aligned} V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n\alpha_n(p(s))\Psi(i,s)ds + (N-n)\int_0^\Delta V_{p^{-i}(s),n+1}\Psi(-i,s)ds + \\ (1 - \sum_j \lim_{s\to\Delta-} \Psi(s,j))[k_n\alpha_n(p(\Delta)) - \beta(1 - p(\Delta))] \end{aligned}$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F^j_{p,S}$. Specifically, for all $s > 0$,

$$\Psi^i(s) = p\lambda \int_0^s e^{-\lambda rn}(1 - F_{p,n}(r))^{N-n} dr$$

$$\Psi^{-i}(s) = p \int_0^s e^{-\lambda r(N-n)}(1-F_{p,n}(r))^{n-2} d(-e^{-\lambda r}(1-F_{p,S}(r)))+(1-p) \int_0^s (1-F_{p,n}(r))^{n-2} dF_{p,n}(r)$$

Note that it follows from Lemma 1 that, for all j, Ψ^j is also absolutely continuous, I.e., there exists a function ψ_j such that:

$$\Psi_j(s) = \int_0^s \psi_j(r)dr.$$

Specifically, according to Lemma 1, one such ψ_i and ψ_{-i} are given by the following:

$$\psi_i(s) = p\lambda e^{-\lambda sn}(1 - F_{p,n}(s))^{N-n}$$

$$\psi_{-i}(s) = pe^{-\lambda sn}(\lambda + F'_{p,n}(s+) - \lambda F_{p,n}(s))(1 - F_{p,n}(s))^{n-2} + (1 - p)(1 - F_{p,n}(s))F'_{p,n}(s+)$$

Substituting these expressions for both $V_{p,n}(\delta_0)$ and $V_{p,n}(\delta_\Delta)$ into (12) and rearranging, we obtain that for all $\Delta \in (0, \varepsilon)$,

$$K_1(\Delta) + K_2(\Delta) + K_3(\Delta) = 0 \tag{13}$$

where

$$K_1(\Delta) \equiv \frac{\int_0^\Delta k_n[(\alpha_n(p(s)) - \alpha_n(p)) + \beta(1 - p)]\psi_i(s)ds}{\Delta}$$

$$K_2(\Delta) \equiv \frac{(N - n)\int_0^\Delta [V_{p^{-i}(s),n+1} - k_n\alpha_n(p) + \beta(1 - p)]\psi_{-i}(s)ds}{\Delta}$$

$$K_3(\Delta) \equiv \frac{(1 - \sum_j \lim_{s\to\Delta-} \psi_j(\Delta))[k_n(\alpha_n(p(\Delta)) - \alpha_n(p)) + \beta(p(\Delta) - p)]}{\Delta}$$

Now, we consider $\lim_{\Delta\to 0+}$ of $K_1(\Delta)$, $K_2(\Delta)$, and $K_3(\Delta)$ separately.

For $K_1(\Delta)$, it follows from L'Hôpital's Rule, together with the continuity of $\alpha_n(p(\Delta))$ (i.e., Lemma 4) and $\psi_i(\Delta)$ in Δ that

$$\lim_{\Delta\to 0+} K_1(\Delta) = \lim_{\Delta\to 0+} [k_n(\alpha_n(p(\Delta)) - \alpha_n(p)) + \beta(1 - p)]\psi_i(\Delta) = \beta(1 - p)\psi_i(0) = \beta(1 - p)p\lambda.$$

For $K_2(\Delta)$, it again follows from L'Hôpital's Rule, together with the right-continuity of

$V_{p^{-i}(\Delta),n+1}$ in Δ that

$$\lim_{\Delta\to 0+} K_2(\Delta) = (N-n)\lim_{\Delta\to 0+}[V_{p^{-i}(\Delta),n+1} - k_n\alpha_n(p) + \beta(1-p)]\psi_{-i}(\Delta)$$
$$= (N-n)[V_{p^{-i},n+1} - k_n\alpha_n(p) + \beta(1-p)](\frac{\lambda p}{\alpha_n(p)})$$

where the final inequality follows from the fact that at all (p,n) on-path, $\alpha_n(p) = \frac{\lambda p}{\lambda p + F'_{p,n}(0)}$.

For $K_2(\Delta)$, first note that by the continuous differentiability of $\Psi_j(s)$ that

$$\lim_{\Delta\to 0+}\sum_j \lim_{s\to\Delta-}\Psi(s,j) = 0.$$

Thus, it follows from the right-differentiability of $\alpha_n(p(\Delta))$ in Δ that

$$\lim_{\Delta\to 0+} K_3(\Delta) = k_n\lim_{\Delta\to 0}\frac{\alpha_n(p(\Delta)) - \alpha_n(p)}{\Delta} + \beta\lim_{\Delta\to 0+}\frac{p(\Delta)-p}{\Delta} = k_n\frac{d}{d\Delta}\alpha_n(p(\Delta))\Big|_{\Delta=0+} + \beta p'(\Delta)\Big|_{\Delta=0+}$$
$$= p'(\Delta)\Big|_{\Delta=0+}[k_n\alpha_n'(p) + \beta] = -\lambda pn(1-p)[k_n\alpha_n'(p) + \beta]$$

Since we have shown that $\lim_{\Delta\to 0+} K_1(\Delta)$, $\lim_{\Delta\to 0+} K_2(\Delta)$, and $\lim_{\Delta\to 0+} K_3(\Delta)$ exist, and are given by the above expressions, it follows from (13) that

$$\lim_{\Delta\to 0+} K_1(\Delta) + \lim_{\Delta\to 0+} K_2(\Delta) + \lim_{\Delta\to 0+} K_3(\Delta) = 0.$$

Substituting in the above expressions for $K_1(\Delta)$, $K_2(\Delta)$ and $K_3(\Delta)$, we obtain (ODE).

Now, we wish to establish that (ODE) must hold whenever $k_n \geq \beta$ or $p > p_n^*$. It follows from Proposition 1 that $\alpha_n(p) < 1$, and thus by the above, (ODE) must hold.

Finally, we establish the two limit conditions presented in the proposition. We begin by establishing that when $k_n \geq \beta$, $\lim_{p\to 0+}\alpha_n(p) = \beta/k_n$. To this end, first note by Lemma 3 that for all $p > 0$, $V_{p,n}(\delta_0) = V_{p,n}(\delta_\infty)$. Note further that

$$\lim_{p\to 0+} V_{p,n}(\delta_\infty) = 0.$$

Thus,

$$\lim_{p\to 0+} V_{p,n}(\delta_0) = \lim_{p\to 0+} k_n\alpha_n(p) - \beta = 0,$$

and therefore, $\lim_{p\to 0+}\alpha_n(p) = \frac{\beta}{k_n}$. Next, let us consider the case where $k_n < \beta$. That $\lim_{p\to p_n^*+}\alpha_n(p) = 1$ follows from Lemma 4, since by Proposition 1, $\alpha_n(p_n^*) = 1$. □

Definition 2. α is a solution to (P) if it satisfies the following for all $n \leq N$ and $p \in (0,1]$:

- If $k_n < \beta$ and $p \leq p_n^* \equiv \frac{k_n - \beta}{k_n/n - \beta}$, then $\alpha_n(p) = 1$.
- If $k_n \geq \beta$ or $p < p_n^*$, then α satisfies (ODE), with limit condition $\lim_{p \to 0+} \alpha_n(p) = \beta / k_n$ if $k_n \geq \beta$ and $\lim_{p \to p_n*+} \alpha_n(p) = 1$ if $k_n < \beta$.
- $\alpha_n(1) = 0$.

Lemma 5. *(α, F) is an equilibrium if and only if at all (p, n) on-path, α is both consistent with F and a solution to (P).*

Proof of Lemma 5. Fix an (α, F). We begin by establishing the necessity of the three conditions specified in Definition 2 for (α, F) to be an equilibrium. First we establish the necessity of the first bullet of Definition 2. To this end, recall that by the selection assumption, $F_{1,n}(0) = 1$. Thus, it follows from (2) that $\alpha_n(1) = 0$ if $(p = 1, n)$ is on-path. Bullets two and three of Definition (2) follow from Proposition 1 and Proposition 2, respectively.

Next, we establish the sufficiency of the above conditions for (α, F) to be an equilibrium. We begin by considering the case in which $k_n < \beta$ and $p \leq p_n^*$. It follows from (P) that $\alpha_n(q) = 1$ for all $q \leq p$. Thus, by (2), $F_{p,n} = \delta_\infty$. We thus wish to show that there exist no profitable deviations in this case, i.e., that $V_{p,n} = V_{p,n}(\delta_\infty)$. It suffices to show that

$$V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s) \text{ for all } s \in [0, \infty). \tag{14}$$

First, note that for all $s \in (0, \infty)$,

$$V_{p,n}(\delta_s) = k_n(1 - p(1 - e^{-\lambda s n})(\frac{N-n}{N-n+1})) - \beta(1-p) \leq k_n - \beta(1-p) = V_{p,n}(\delta_0).$$

Further, $k_n \leq \beta$ and $p \leq p_n^*$ implies that

$$V_{p,n}(\delta_0) = k_n - \beta(1-p) \leq \frac{k_n}{n} = V_{p,n}(\delta_\infty)$$

Thus, $V_{p,n}(\delta_\infty) \geq V_{p,n}(\delta_s)$ for all $s \in [0, \infty)$

Next, we show that $F_{p,n}$ is optimal when $k_n \geq \beta$ or $p < p_n^*$. To this end, we begin by showing that

$$\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) = 0 \text{ for all } \Delta \in [0, \infty) \text{ if } k_n \geq \beta \text{ and for all } \Delta \in [0, t^*) \text{ if } k_n < \beta \tag{15}$$

where t^* is the unique value such that $p(t^*) = p_n^*$. Note that

$$V_{p,n}(\delta_\Delta) = \int_0^\Delta k_n \alpha_n(p(s)) d\Psi^i(s) + \int_0^\Delta V_{p^i(s),n+1} d\Psi^{-i}(s) + \\ (1 - \sum_j \Psi^j(\Delta))(\alpha_n(p(\Delta)) - \beta(1 - p(\Delta))) \tag{16}$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,n}$. Then, it follows that

$$\begin{aligned}
&\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) \\
&= k_n \alpha_n(p(\Delta)) \Psi^{i\prime}(\Delta) + (N-n) V_{p^i(\Delta),n+1} \Psi^{-i\prime}(\Delta) + (1 - \sum_j \Psi^j(\Delta)) p'(\Delta)[\alpha_n'(p(\Delta)) - \beta] \\
&- \sum_j \Psi^{j\prime}(\Delta)(k_n \alpha_n(p(\Delta)) - \beta(1 - p(\Delta))) \\
&= (N-n)[V_{p^i(\Delta),n+1} - k_n \alpha_n(p(\Delta)) + \beta(1 - p(\Delta))] \Psi^{-i\prime}(\Delta) - \beta(1 - p(\Delta)) \Psi^{i\prime}(\Delta) \\
&+ (1 - \sum_j \Psi^j(\Delta)) p'(\Delta)(k_n \alpha'(p(\Delta), n) - \beta)
\end{aligned}$$

Note that in the above, the existence of $\Psi^{j\prime}(\Delta)$ follows from the differentiability of α_n at $p(\Delta)$, and thus, the differentiability of $F_{p,n}$ at Δ. We wish to show that $\frac{d}{d\Delta} V_{p,n}(\delta_\Delta) = 0$. To this end, we begin by deriving expressions for $\Psi^{i\prime}(\Delta)$ and $\Psi^{-i\prime}(\Delta)$. First, it follows by definition of the first-report distribution that:

$$\Psi^i(\Delta) = p\lambda \int_0^\Delta (1 - F_{p,n}(s))^{N-n} e^{-\lambda n s} ds.$$

Differentiating this, we obtain:

$$\Psi^{i\prime}(\Delta) = p\lambda(1 - F_{p,n}(\Delta))^{N-n} e^{-\lambda n \Delta}$$

Meanwhile:

$$\Psi^{-i}(\Delta) = p \int_0^\Delta (1 - F_{p,n}(s))^{n-2} e^{-\lambda(N-n)s} d((F_{p,n}(s) - 1) e^{-\lambda s}) + (1-p) \int_0^\Delta (1 - F_{p,n}(s))^{n-2} F_{p,n}'(s) ds$$

where the existence of $F_{p,n}'(s)$ again follows from the assumption that α_n is differentiable at $p(s)$. Differentiating this, we obtain:

$$\Psi^{-i\prime}(\Delta) = p(1 - F_{p,n}(\Delta))^{n-2} e^{-\lambda\Delta n}[F'_{p,n}(\Delta) + \lambda(1 - F_{p,n}(\Delta))] + (1-p)(1 - F_{p,n}(\Delta))^{n-2} f_{p,n}(\Delta)$$
$$= (1 - F_{p,n}(\Delta))^{N-n}[\frac{f_{p,n(\Delta)}}{1 - F_{p,n}(\Delta)}(pe^{-\lambda\Delta n} + (1-p)) + pe^{-\lambda\Delta n}\lambda]$$

It follows from the definition of α (equation (2)) and the consistency condition (equation (1)) that

$$\frac{F'_{p,n}(\Delta)}{1 - F_{p,n}(\Delta)} = \lambda p(\Delta)(\frac{1}{\alpha_n(p(\Delta))} - 1).$$

Substituting this, along with the definition of $p(\Delta)$ (equation (4)), we obtain:

$$\Psi^{-i\prime}(\Delta) = \lambda(1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta n} + (1-p))\frac{p(\Delta)}{\alpha_n(p(\Delta))}$$

Note further that

$$1 - \sum_j \Psi^j(\Delta) = (1 - F(\Delta))^{N-n}(pe^{-\lambda\Delta n} + (1-p)) \tag{17}$$

Substituting equations the expressions for $\Psi^{i\prime}(\Delta)$, $\Psi^{-i\prime}(\Delta)$, and $1 - \sum_j \Psi^j(\Delta)$ into the above equation for $\frac{d}{d\Delta}V_{p,n}(\delta_\Delta)$, and simplyifying, we obtain:

$$\frac{d}{d\Delta}V_{p,n}(\delta_\Delta) = K[\frac{(N-n)}{\alpha_n(p(\Delta))}(V^i(p(\Delta), n+1) - k_n\alpha_n(p(\Delta)) + \beta(1 - p(\Delta))(1 - \alpha_n(p(\Delta))))$$
$$-k_n\alpha'(p(\Delta), n)(1 - p(\Delta))n]$$

where $K \equiv \lambda(1 - F_{p,n}(\Delta))^{N-n}(pe^{-\lambda\Delta n} + (1-p))p(\Delta)$. Because (ODE) is satisfied at $(p(\Delta), n)$, using it to substitute in for $\alpha'(p(\Delta, n)$, we obtain (15).

Now, consider the case where $k_n \geq \beta$. To show $F_{p,n}$ is optimal, it suffices to show that all pure strategies δ_Δ yield the same payoff, i.e., that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \tag{18}$$

for all $\Delta \in [0, \infty]$. It follows directly from (15) that (18) holds for all $\Delta \in [0, \infty)$. It remains to show that (18) holds for $\Delta = \infty$. To this end, first note that by (15),

$$\begin{aligned}
V_{p,n}(\delta_0) &= \lim_{\Delta\to\infty} V_{p,n}(\delta_\Delta) \\
&= \lim_{\Delta\to\infty} \int_0^\Delta k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\lim_{\Delta\to\infty}\int_0^\Delta V(p^i(s), n+1)d\Psi^{-i}(s) + \\
&\lim_{\Delta\to\infty}(1-\sum_j \Psi^j(\Delta))(k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta))) \\
&= \int_0^\infty k_n\alpha_n(p(\Delta))d\Psi^i(s) + (N-n)\int_0^\infty V(p^i(\Delta), n+1)d\Psi^{-i}(s) = V_{p,n}(\delta_\infty)
\end{aligned}$$

where the third equality follows from the limit condition $\lim_{p\to 0+}\alpha(p,n) = \beta/k_n$:

$$\lim_{\Delta\to\infty} k_n\alpha_n(p(\Delta)) - \beta(1-p(\Delta)) = \lim_{p\to 0+} k_n\alpha_n(0) - \beta = 0.$$

Finally, consider the case where $k_n < \beta$ and $p > p_n^*$. Note that because $\alpha_n(p(s)) = 1$ for all $s > t^*$, by (2), it follows that $F'_{p,n}(s) = 0$ for all such s. It follows from this that the support of $F_{p,n}$ lies within $[0,t^*]\cup\infty$. Thus, to show $F_{p,n}$ is optimal, it suffices to show that δ_Δ is optimal for $\Delta \in [0,t^*]\cup\infty$. To this end, we will proceed by first showing

$$V_{p,n}(\delta_\Delta) = V_{p,n}(0) \text{ for all } \Delta \in [0,t^*]\cup\infty \tag{19}$$

and then showing

$$V_{p,n}(\delta_{t^*}) \geq V_{p,n}(\delta_\Delta)\text{for all } \Delta \in (t^*,\infty). \tag{20}$$

To show (19), first recall that it follows from (15) that

$$V_{p,n}(\delta_0) = V_{p,n}(\delta_\Delta) \text{ for all } \Delta \in [0,t^*).$$

It remains to show $V_{p,n}(\delta_0) = V_{p,n}(\delta_s)$ for $s \in \{t^*,\infty\}$. For $s = t^*$, note that it follows from the above that

$$V_{p,n}(\delta_0) = \lim_{\Delta\to t^*-} V_{p,n}(\delta_\Delta) = V_{p,n}(\delta_{t^*})$$

where the final inequality follows from (16), observing that α_n is continuous at p_n^* and Ψ^j is continuous at t^*. We will now show $V_{p,n}(\delta_{t^*}) = V_{p,n}(\delta_\infty)$. To this end, note that for all $\Delta \in [t^*,\infty]$:

$$V_{p,n}(\delta_\Delta) = \int_0^{t^*} k_n\alpha_n(p(s))d\Psi^i(s) + (N-n)\int_0^{t^*} V(p^i(s), n+1)d\Psi^{-i}(s) + (1-\sum_j \Psi^j(t^*))V(p_n^*, n, \delta_{\Delta-t^*})$$

Thus, to show $V(p, n, \delta_t^*) = V_{p,n}(\delta_\infty)$, it suffices to show that $V(p_n^*, n, \delta_0) = V(p_n^*, n, \delta_\infty)$. But it follows from the definition of p_n^* that:

$$V(p_n^*, n, \delta_0) = k_n - \beta(1 - p_n^*) = \frac{k_n p_n^*}{n} = V(p_n^*, n, \delta_\infty).$$

Similarly, to show (20), it suffices to show that $V(p_n^*, n, \delta_0) \geq V(p_n^*, n, \delta_\Delta)$ for all $\Delta \in (0, \infty)$, which we had previously established in (14). □

Proof of Theorem 1. Fix an n. Assume inductively that there exists a unique solution to (P) for all $m < n$. We wish to show that ther exists a unique solution to (P) for n. To establish this, it suffices to show there exists a unique solution to the following two limit problems, when $\beta \leq k_n$ and $\beta > k_n$, respectively:

$$\text{(ODE) is satisfied on } (0,1), \text{ and } \lim_{p \to 0+} \alpha_n(p) = \beta / k_n \qquad \text{(LP: } \beta \leq k_n\text{)}$$

$$\text{(ODE) is satisfied on } (0, p^*), \text{ and } \lim_{p \to p_n^*+} \alpha_n(p) = 1. \qquad \text{(LP: } \beta > k_n\text{)}$$

To establish existence and uniqueness two the two above problems, we proceed by extending them to two boundary value problems. To this end, we begin by defining an extension of (ODE′) of (ODE), which is identical to (ODE), except that it is well-defined when $p^i \geq 1$. Specifically, define:

$$\alpha_n'(p) = -\frac{1}{k_n(1-p)\alpha_n(p)} \frac{N-n}{N-n+1} [k_n \alpha_n(p) - \tilde{V}(p^i, n+1) - \beta(1 - \alpha_n(p))(1-p)] \quad \text{(ODE′)}$$

where

$$\tilde{V}(p^i, n+1) = \begin{cases} V(p^i, n+1) & \text{if } p^i \in (0,1) \\ 0 & \text{if } p^i \geq 1 \end{cases}$$

Now let us define two boundary value problems on (ODE′):

$$\text{(ODE′) is satisfied on } [0,1), \text{ and } \alpha_n(0) = \beta / k_n \qquad \text{(BVP: } \beta \leq k_n\text{)}$$

$$\text{(ODE) is satisfied on } (0, p_n^*], \text{ and } \alpha_n(p^*) = 1. \qquad \text{(BVP: } \beta \geq k_n\text{)}$$

Now we claim that the existence and uniqueness of a solution to (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$) implies the existence and uniqueness of a solution to (LP: $\beta \leq k_n$) and (LP: $\beta > k_n$), respectively. Let us begin by considering the case where $k_n \geq \beta$. Assume that there exists a unique solution α_n to (BVP: $\beta \le k_n$). Note that order for α_n to satisfy (BVP: $\beta \le k_n$), it must be that $\lim_{p\to 0+} \alpha_n(p) = k_n/\beta$. Furthermore, (ODE) and (ODE′) are equivalent on $(0, 1)$. It follows that α_n is a solution to (LP: $\beta \le k_n$), thus establishing existence. To establish uniqueness, assume by contradiction there exists some $\tilde{\alpha}_n$ defined on $p \in (0, 1)$ that is a solution to (LP: $\beta \le k_n$) where $\tilde{\alpha}_n(p) \ne \alpha_n(p)$. Now, define $\hat{\alpha}_n$ as follows, which extends the domain of $\tilde{\alpha}_n$:

$$\hat{\alpha}_n(p) = \begin{cases} \tilde{\alpha}_n(p) & \text{if } p \in (0,1) \\ k_n/\beta & \text{if } p = 0 \end{cases}$$

Because $\lim p \to 0+ \tilde{\alpha}_n(p) = k_n/\beta$, it follows that $\hat{\alpha}_n(p)$ satisfies (ODE′) on $p \in [0, 1]$ and is thus a solution to (BVP: $\beta \le k_n$). Thus, (BVP: $\beta \le k_n$) does not have a unique solution, a contradiction. Note that the argument in the case where $k_n < \beta$ is analogous.

It remains for us to establish that there exist unique solutions to both (BVP: $\beta \le k_n$) and (BVP: $\beta \ge k_n$). We do this by invoking the Picard existence and uniqueness theorem, and thus begin by establishing that the right-hand side of (ODE′) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p for $p \in [-\varepsilon, 1)$ and $\alpha_n(p) \in [c, 1+\varepsilon]$ for any $c > 0$ and some $\varepsilon > 0$. Since $p^i \equiv \alpha_n(p) + (1 - \alpha_n(p))p$, it suffices to show that $\tilde{V}(\cdot, n+1)$ is Lipschitz continuous in p^i for $p^i \ge 0$. In the case where $n = 1$, $\tilde{V}(p^i, n+1) = 0$ for all p^i, and this is immediate. Next, suppose $n > 1$. First, consider the case where $k_{n+1} \ge \beta$. It follows from Lemma 3 that:

$$\tilde{V}(p^i, n+1) = \begin{cases} k_n \alpha_{n+1}(p^i) - \beta(1 - p^i) & \text{if } p^i < 1 \\ 0 & \text{if } p^i > 1 \end{cases}$$

Because $\tilde{V}(p^i, n+1)$ is continuously differentiable in p^i when $p^i \ne 1$, to establish that it is Lipschitz continuous it suffices to show that $\lim_{p^i \to 1-} V(p^i, n+1) = 0$. Suppose this does not hold, by contradiction. Because $\alpha_{n+1}(\cdot)$ satisfies (ODE), this implies that $\lim_{p^i \to 1-} \alpha'_{n+1}(p^i) = \infty$. This in turn implies that $\lim_{p^i \to 1} \alpha_{n+1}(p^i) = \infty$, and thus that (ODE) is not satisfied at $p^i = 1$. Contradiction.

Next, consider the case where $k_{n+1} < \beta$. In this case:

$$\tilde{V}(p^i, n+1) = \begin{cases} k_{n+1} p^i / n & \text{if } p^i < p^*_{n+1} \\ k_n \alpha_{n+1}(p^i) - \beta(1 - p^i) & \text{if } p^i \in (p^*_{n+1}, 1) \\ 0 & \text{if } p^i = 1 \end{cases}$$

By the reasoning from the case where $k_{n+1} \ge \beta$, $\tilde{V}(p^i, n+1)$ is Lipschitz continuous for all $p^i > p^*_{n+1}$. Furthermore, Lipschitz continuity holds on $p^i < p^*_{n+1}$. To show that Lipschitz continuity holds across all p^i, it suffices to show that $\tilde{V}(\cdot, n+1)$ is differentiable at p^*_{n+1}. To this end, we take the left- and right- derivative of $\tilde{V}(\cdot, n+1)$ at p^*_{n+1} and show that they are equal:

$$\tilde{V}_1(p^*-, n+1) = \frac{k_{n+1}}{N-n}$$

$$\tilde{V}_1(p^*+, n+1) = -k_{n+1}\alpha'_{n+1}(p^*_{n+1}) + \beta = \frac{k_{n+1}}{1-p^*_{n+1}} \frac{N-n}{N-n+1} - \beta = \frac{k_{n+1}}{N-n}$$

Now, we show that there exists a unique solution for both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$) in some neighborhood of their respective boundary conditions. By the Picard Theorem, this follows immediately from our above-established result that the right-hand side of (ODE) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p in some neighborhood of the boundary conditions $(\alpha_n(p) = 1, p = p^*)$ and $(\alpha_n(p) = \beta/k_n, p = 0)$.

Next, we seek to establish global existence and uniqueness of solutions to both (BVP: $\beta \leq k_n$) and (BVP: $\beta \geq k_n$). First, consider (BVP: $\beta \geq k_n$). The argument for (BVP: $\beta \leq k_n$) follows analogously. Let $[p^*, \overline{p})$ denote the largest right-open interval such that existence and uniqueness are both satisfied. Assume by contradiction that $\overline{p} < 1$. Let $\alpha_n(p)$ denote the solution along this interval.

We begin by showing that on this interval, $\alpha_n(p) \in (\underline{\alpha}, 1]$, where $\underline{\alpha} > 0$ is some constant. The upper bound is established as follows: suppose by contradiction that $\alpha_n(p) > 1$ somewhere on the interval. By the continuous differentiability of α_n along the interval, there must exist some $q < p$ such that $\alpha_n(q) = 1$ and $\alpha'_n(q) \geq 0$. However, it follows from (ODE′) that

$$\alpha'_n(q) = -\frac{1}{k_n(1-q)} \frac{N-n}{N-n+1}[k_n - \tilde{V}(p^i, n+1)] < 0$$

where the strict inequality follows from the fact that $\tilde{V}(p^i, n+1) \leq k_{n+1} < k_n$. Contradiction. The lower bound is established as follows: suppose by contradiction that such a lower bound does not exist. Then, again by the continuous differentiability of α_n along the interval, there exists some $\hat{p} \in [p^*, \overline{p})$ such that

$$\lim_{p \to \hat{p}-} \alpha_n(p) = 0 \text{ and } \alpha_n(p) > 0 \text{ for all } p < \hat{p}$$

However, it then follows from (ODE) that $\lim_{p\to\hat{p}-} \alpha'_n(p) = \infty$. Thus, (ODE′) is not satisfied on $[p^*, \overline{p})$. Contradiction.

Having established that on $[p^*, \overline{p})$, $1 \leq \alpha_n(p) > \underline{\alpha} > 0$, it follows from (ODE′), and the observation that $V(p^i, n+1)$ is bounded, that α'_n is also bounded on this range. Thus, it follows that $\lim_{p\to\overline{p}-} \alpha_n(p) \equiv \overline{\alpha} > 0$ exists.

Now, consider the following modified boundary value problem, which is identical to (BVP: $\beta \geq k_n$), except with boundary condition $(\overline{p}, \overline{\alpha})$. by our prior-established result, we recall that (ODE′) is Lipschitz continuous in $\alpha_n(p)$ and continuous in p in some neighborhood of the boundary condition. Thus, we can again apply the Picard Theorem to obtain that there exists a unique solution to the modified boundary value problem in some neighborhood of $(\overline{p}, \overline{\alpha})$. Formally, there exists some $\varepsilon > 0$ such that there is a unique solution $\tilde{\alpha}_n(p)$ on interval $(\overline{p} - \varepsilon, \overline{p} + \varepsilon)$. Now, we can "paste" this solution $\tilde{\alpha}_n$, with our prior solution α_n. Formally, let

$$\hat{\alpha}_n(p) = \begin{cases} \alpha_n(p) & \text{if } p \in [p_n^*, \overline{p}) \\ \tilde{\alpha}_n(p) & \text{if } p \in [\overline{p}, \overline{p} + \varepsilon) \end{cases}$$

Now, note that $\hat{\alpha}_n(p)$ is a unique solution to (BVP: $\beta \geq k_n$) on $[p_n^*, \overline{p}+\varepsilon)$, which contradicts our earlier assumption that $[p^*, \overline{p})$ was the largest right-open interval such that existence and uniqueness are satisfied. Contradiction. □

Proof of Proposition 3. Let us begin by showing that $\alpha_n(p)$ is decreasing in p for all (p, n) on-path. When by Lemma 5, it follows that when $k_1 < \beta$, $\alpha_1(p) = 1$ for all p, and otherwise, $\alpha_1'(p) = 0$ for all p. Thus we have shown that $\alpha_1(p)$ is constant in p. Now, consider the case where $n \geq 2$. Assume inductively that $\alpha_{n+1}(p)$ is weakly decreasing in p whenever $(p, n+1)$ is on path.

Assume by contradiction that there exists some $\overline{p}$ such that α_n is strictly increasing. Note that Lemma 5, $\alpha_n'(p) = 0$ whenever $\beta \geq k_n$ and $p < p_n^*$. Thus it must be that $\beta > k_n$ or $\overline{p} > p_n^*$. In this case, it again follows from Lemma 5 that (ODE) must be satisfied. Now define the function $X(p)$ as follows:

$$X(p) \equiv k_n \alpha_n(p) - \beta(1 - p^i) - V(p^i, n + 1) \tag{21}$$

Note that whenever (ODE) is satisfied, the following holds:

$$\alpha_n'(p) > (=) 0 \text{ if and only if } X(p) < (=) 0 \tag{22}$$

Thus, $X(\overline{p}) < 0$. Now, I claim that there exists $\underline{p} < \overline{p}$ such that $\lim_{p \to \underline{p}+} X(p) \geq 0$. To establish this, first consider the case where $k_n \geq \beta$. In this case,

$$\lim_{p \to 0+} X(p) = k_n \lim_{p \to 0+} \alpha_n(p) - \beta(1 - \lim_{p \to 0+}) - \lim_{p \to 0+} V(\alpha_n(p), n+1) = (k_n + \beta) \lim_{p \to 0+} \alpha_n(p) - \beta - \lim_{p \to 0+} V(\alpha_n(p), n+1)$$

When $\lim_{p \to 0+} \alpha_{n+1}(\alpha_n(p)) < 1$, it follows from Lemma 3 that

$$\lim_{p \to 0+} V(\alpha_n(p), n+1) = \lim_{p \to 0+} V(\alpha_n(p), n+1, \delta_0) = k_{n+1} \lim_{p \to 0+} \alpha_{n+1}(\alpha_n(p)) - \beta(1 - \lim_{p \to 0+} \alpha_n(p))$$
$$= k_{n+1}\alpha_{n+1}(\beta/k_n) - \beta(1 - \beta/k_n)$$

Note that because $k_n \geq \beta$, the final equality from Lemma 5. Substituting this into our above expression for $\lim_{p \to 0+} X(p)$, we obtain

$$\lim_{p \to 0+} X(p) = \beta - k_{n+1}\alpha_{n+1}(\beta/k_n)$$

In the case where $k_{n+1} < \beta$, it follows directly that $\lim_{p \to 0+} X(p) > 0$. Otherwise, if $k_{n+1} \geq \beta$, then because $\lim_{p \to 0+} \alpha_{n+1}(p) = \beta/k_{n+1}$, it follows from the inductive assumption that $\alpha_{n+1}(p) \leq \beta/k_{n+1}$ for all p, and thus that $\lim_{p \to 0+} X(p) > 0$.

Meanwhile, when $\lim_{p \to 0+} \alpha_{n+1}(\alpha_n(p)) = 1$, it follows from the inductive assumption that $\alpha_{n+1}(q) = 1$ for all $q \geq \lim_{p \to 0+} \alpha_n(p)$. It thus follows that

$$\lim_{p \to 0+} V(p^i, n+1) = \lim_{p \to 0+} V(p^i, n+1, \delta_\infty) = \frac{k_{n+1}}{N-n}\frac{\beta}{k_n}$$

Substituting into the above expression for $\lim_{p \to 0+} X(p)$ and simplifying, we obtain

$$\lim_{p \to 0+} X(p) = (\beta/k_n)(\beta - k_{n+1}/(N-n)) \geq 0,$$

where the inequality follows from the fact that $\alpha_{n+1}(\beta/k_n) = 1$, implying by Lemma 5 that $k_{n+1} \geq \beta$.

Next, consider the case where $k_n < \beta$. In this case,

$$\lim_{p \to p_n^* +} X(p) = k_n - \lim_{p^i \to 1-} V(p^i, n+1)$$

If $\lim_{p^i \to 1-} \alpha_{n+1}(p^i) < 1$, then by Lemma 3,

$$\lim_{p^i \to 1-} V(p^i, n+1) = \lim_{p^i \to 1-} V(p^i, n+1, \delta_0) = k_{n+1} \lim_{p^i \to 1-} \alpha_{n+1}(p^i) < k_n.$$

Thus, in this case, we obtain that $\lim_{p \to p_n^* +} X(p) > 0$. Meanwhile, if $\lim_{p^i \to 1-} \alpha_{n+1}(p^i) = 1$, then by the inductive assumption, $\alpha_{n+1}(p) = 1$ for all p. Thus,

$$\lim_{p^i \to 1-} V(p^i, n+1) = \lim_{p^i \to 1-} V(p^i, n+1, \delta_\infty) = \lim_{p^i \to 1-} \frac{k_{n+1}p^i}{N-n} = \frac{k_{n+1}}{N-n}.$$

We once again obtain $\lim_{p \to p_n^*+} X(p) > 0$. We have thus completed showing that there always exists $\underline{p} < \overline{p}$ such that $\lim_{p \to \underline{p}+} X(p) \geq 0$.

Because $X(\overline{p}) < 0$ by assumption, there must exist some $q \in [\underline{p}, \overline{p}]$ $X(q) < 0$ and $X'(q) < 0$. Note that differentiating our above expression for X, we have

$$X'(q) = k_n \alpha_n'(q) + \beta((1-q)\alpha_n'(q) + (1-\alpha_n(q))) - \frac{d}{dq} V(\alpha_n(q) + (1-\alpha_n(q))q). \tag{23}$$

First, consider the case where $\alpha_{n+1}(q^i) < 1$. Then by Lemma 3,

$$V(q^i, n+1) = V(q^i, n+1, \delta_0) = k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i).$$

Substituting this into (23), we obtain

$$X'(q) = k_n \alpha_n'(q) - k_{n+1}\alpha_{n+1}'(q^i)((1-q)\alpha_n'(q) + (1-\alpha_n(q))).$$

Note that because $X(q) < 0$ it follows from (22) that $\alpha_n'(q) > 0$. Furthermore, by the inductive assumption, $\alpha_{n+1}'(q^i) \leq 0$. Thus, in this case, $X'(q) > 0$. Contradiction.

Next, consider the case where $\alpha_{n+1}(q^i) = 1$. By the inductive assumption, $\alpha_{n+1}(p) = 1$ for all $p \leq q^i$. Thus,

$$V(q^i, n+1) = V(q^i, n+1, \delta_\infty) = \frac{k_{n+1} q^i}{N-n}.$$

Substituting this into (23), we have

$$X'(q) = k_n \alpha_n'(q) + (\beta - \frac{k_{n+1}}{N-n})((1-q)\alpha_n'(q) + (1-\alpha_n(q)))$$

Because $\alpha_{n+1}(q^i) = 1$, by Proposition 1 (if $n > 2$) and Lemma 5 (if $n = 2$), it must be that $\beta \geq k_{n+1}$. Thus, it must be that $X'(q) > 0$. Contradiction.

Next, we will show that if $k_1 \geq \beta$, then $\alpha_n(p) = \beta/k_n$. Assume that $k_1 \geq \beta$. First consider the case where $n = 1$. By Lemma 5, $\alpha_n'(p) = 0$ for all p on-path, and thus, $\alpha_1(p)$ is constant in p. Since Lemma 5 also asserts that $\lim_{p \to 0+} k_1 \alpha_1(p) = \beta$, it must be that $\alpha_1(p) = \beta/k_1$ for all p. Now, consider $n > 1$. Assume inductively that $\alpha_{n+1}(p) = \beta/k_{n+1}$ at all p. We begin by showinggmail that $\alpha_n(p)$ is constant in p. Since $k_n > \beta$, by Lemma 5, (ODE) must hold at all p. By (22)showing $\alpha_n(p)$ is constant in p is equivalent to showing that $X(p) = 0$. To establish this, I begin by claiming that $V(p^i, n+1) = V(p^i, n+1, \delta_0)$. In the case where $k_{n+1} > \beta$, it follows from that $\alpha_{n+1}(p^i) < 1$, and thus must hold by Lemma 3. In the case where $k_{n+1} = \beta$, because for $k_m > k_1 \geq \beta$ for all $m \geq 2$, it follows that $n + 1 = N$. In this case, all pure strategies δ_s must yield the same value. In particular, for all $s \in [0, \infty]$, $V(p, N, \delta_s) = k_1 p$. Thus, δ_0 must be trivially optimal. Having established that $V(p^i, n+1) = V(p^i, n+1, \delta_0)$, we have:

$$V(p^i, n+1) = k_{n+1}\alpha_{n+1}(p^i) - \beta(1 - p^i) = \beta p^i$$

Substituting this into (21), we obtain $X(p) = k_n\alpha_n(p) - \beta$. Now, note that because $\lim_{p\to 0+} \alpha_n(p) = k_n/\beta$. Since we established above that $\alpha_n(p)$ is weakly decreasing, $\alpha_n(p) \leq k_n/\beta$ for all p, and thus $X(p) \leq 0$. Separately, by (22) $\alpha_n(p)$ weakly decreasing implies that $X(p) \geq 0$. Combining these, inequalities, we have $X(p) = 0$.

Finally, we will show that $k_1 < \beta$ implies that $\alpha'(p) < 0$ whenever $\alpha_n(p) < 1$. To this end, suppose $k_1 < \beta$, and suppose by contradiction that at some q such that $\alpha_n(q) < 1$, $\alpha_n'(q) = 0$. It follows from 22 that $X(q) = 0$.

First, consider the case where $\alpha_{n+1}(q^i) = 1$. Recall from the above that in this case, we have

$$X'(q) = k_n\alpha_n'(q) + (\beta - \frac{k_{n+1}}{N-n})((1-q)\alpha_n'(q) + (1 - \alpha_n(q)) = (\beta - \frac{k_{n+1}}{N-n})(1 - \alpha_n(q)) \quad (24)$$

Now, I claim that $\beta > \frac{k_{n+1}}{N-n}$. In the case where $n = 2$, this follows directly from our assumption that $k_1 < \beta$. Meanwhile, in the case where $n = 2$, $\alpha_{n+1}(q^i) = 1$, this follows from Proposition 1. It thus follows from (24) that $X'(q) > 0$. Since $X(q) = 0$, it follows that for some $p < q$, we must have $X(p) < 0$. By (23) it then follows that $\alpha_n'(p) > 0$. This is a a contradiction of our above-established assertion that $\alpha_n(p)$ is weakly decreasing in p.

Next, consider the case where $\alpha_{n+1}(q^i) < 1$. As we have established above, in this case,

$$X'(q) = k_n\alpha_n'(q) - k_{n+1}\alpha_{n+1}'(q^i)[(1-q)\alpha_n'(q) + (1 - \alpha_n(q))] = -k_{n+1}\alpha_{n+1}'(q)[1 - \alpha_n(q)] > 0.$$

Again, this implies that there exists some $p < q$ such that $X(p) < 0$ and thus that $\alpha'(p) > 0$. Contradiction. □

Appendix F Commitment solution

Here, we seek the optimal solution to the monopoly case of the baseline model in which the firm has the ability to *commit* to a reporting strategy. Formally, the only modification we introduce is that rather than F and α being determined simultaneously as they are in equilibrium, the firm chooses its strategy F *before* α is determined. Thus, in the commitment case, the credibility function is *a function* of the firm's strategy. We formalize this dependence by denoting the firm's credibility function as α_F. α_F is then given by (2) as in the non-commitment case, except that the strategy F upon which it is computed is the firm's choice of strategy, rather than the equilibrium strategy.

The firm's objective is to choose a permissible strategy $F \in \mathcal{F}$ which maximizes its expected payoff over the course of the game. Specifically, its problem is given by the following:

$$\max_{F \in \mathcal{F}} \int_0^\infty [\alpha_F(t) - \beta(1-p(t))(1-\alpha(t))] d\Psi(t) \tag{25}$$

where, as in the baseline setup, $\Psi(t)$ denotes probability that the firm reports before time t under strategy F. Before proceeding, we highlight that the only difference between this problem and the problem of the monopoly case of the baseline model is that the credibility function is not taken as given, but is rather a function of the firm's choice of strategy F.

In the analysis that follows, it will be useful for us to cast this problem as a choice of an optimal credibility function α, rahter than an optimal strategy F. In order to do so, we begin with a useful observation, which is analogous to Lemma 1, except under the commitment setting:

Lemma 6. *F must be continuous in equilibrium.*

We omit a proof for this claim, as it follows analogously to the proof for Lemma 1. The proof is analogous because its underlying reasoning is identical to the non-commitment case: if F exhibits a discontinuity at some time t, reporting at this time must yield a negative expected payoff. Thus, the firm can profitably deviate by shifting that it had placed on t to ∞.

It follows immediately from Lemma 6 that in equilibrium, both the firm's strategy F and the corresponding commitment function, α_F, are fully defined by the right-hazard rate $b(t)$ of the firm's strategy. That is the case of α_F, we have

$$\alpha_F(t) = \frac{\lambda p(t)}{\lambda p(t) + b(t)}$$

It further follows that Ψ is continuous and can thus be written as a function fo α_F as follows:

$$\Psi(t) = 1 - e^{-\int_0^t (b(s)+p(s)\lambda)ds} = 1 - e^{-\int_0^t \frac{\lambda p(s)}{\alpha(s)} ds}$$

Having written Ψ in terms of α_F, and noting that at any given t $\alpha_F(t)$ is a one-to-one function of $b(t)$, we can recast the optimization problem given by (25) as one one over α_F:

$$\max_{\alpha_F} \int_0^\infty \lambda p(t)[1 - \beta(1-p(t))(\frac{1}{\alpha_F(t)} - 1)] e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)} ds}$$

In the following claim, we show that the optimal strategy for the firm consists of truth telling always (i.e., $\alpha_F(t) = 1$ for all t). In the proof that follows, we will let $V(t, \alpha_F)$ denote the firm's value at time t given that it has chosen α_F.

Proposition 6. *In equilibrium, $\alpha_F(t) = 1$ for all t.*

Proof. Assume not, by contradiction. Then there exists a t^* such that $\alpha(t^*) < 1$. It follows from Lemma 6, and the assumption that F is right-continuously differentiable, that α_F must be right-continuous. Thus, there must exist a $\hat{\alpha} < 1$ and $\varepsilon > 0$ such that $\alpha_F(t) < \hat{\alpha}$ for all $t \in [t^*, t^* + \varepsilon]$.

Note that for any α_F, including the equilibrium α_F, we can write the time-0 value as follows:

$$V(0, \alpha_F) = \int_0^{t^*+\varepsilon} \lambda p(t)[1 - \beta(1 - p(t))](\frac{1}{\alpha_F(t)} - 1)e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)} ds} dt + e^{-\int_0^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)} ds} V(t^* + \varepsilon, \alpha_F) \tag{26}$$

Now, consider the following deviation $\tilde{\alpha}_F$, which is identical to α_F, except that it is 1 on the interval $[t^*, t^* + \varepsilon]$:

$$\tilde{\alpha}_F(t) = \begin{cases} 1 & \text{if } t \in [t^*, t^* + \varepsilon] \\ \alpha_F(t) & \text{otherwise} \end{cases}$$

Now, it follows from (26) that

$$\begin{aligned} V(0, \alpha_F) = V(0, \tilde{\alpha_F}) + \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1 - \beta(1 - p(t))(\frac{1}{\alpha_F(t)} - 1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha(s)} ds} dt - \int_{t^*}^{t^*+\varepsilon} \lambda p(t) e^{-\int_0^t \lambda p(s) ds} dt \\ +(e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha(s)} ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s) ds}) V(t^* + \varepsilon, \alpha_F) \end{aligned} \tag{27}$$

Now, we will note the following two inequalities:

$$\begin{aligned} \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1 - \beta(1 - p(t))(\frac{1}{\alpha_F(t)} - 1)]e^{-\int_0^t \frac{\lambda p(s)}{\alpha_F(s)} ds} dt \leq \int_{t^*}^{t^*+\varepsilon} \lambda p(t)[1 - \beta(1 - p(t))(\frac{1}{\overline{\alpha}} - 1)]e^{-\int_0^t \frac{\lambda p(s)}{\overline{\alpha}} ds} dt \\ < \int_{t^*}^{t^*+\varepsilon} \lambda p(t) e^{-\int_0^t \lambda p(s) ds} dt \end{aligned}$$

$$e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\alpha_F(s)} ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s) ds} \leq e^{-\int_{t^*}^{t^*+\varepsilon} \frac{\lambda p(s)}{\overline{\alpha}} ds} - e^{-\int_{t^*}^{t^*+\varepsilon} \lambda p(s) ds} < 0$$

Applying these two inequalites to (27) we obtain

$$V(0, \alpha_F) < V(0, \tilde{\alpha}),$$

and thus, $\tilde{\alpha}_F$ serves as a profitable deviation. Contradiction. □

Appendix G Comparative statics: proofs

Proof of Proposition 4. First, let us show part (a). Fix all other parameters and let $0 < \beta < \tilde{\beta}$. Let α and $\tilde{\alpha}$ denote the equilibrium credibility functions under β and $\tilde{\beta}$, respectively. Fix an n and assume inductively that the proposition holds for $n+1$ if $n < N$. Note that for any (p, n) and t, $p(t)$ will be the same $p(t)$ will be the same under β and $\tilde{\beta}$. Thus to show the above claim, it suffices to show that for any p, $\alpha_n(p)$ is weakly increasing in β, and strictly so whenever $\alpha_n(p) < 1$.

We begin by showing that $\alpha_n(p) = 1$ implies that $\tilde{\alpha}_n(p) = 1$. First, consider the case where $n = 1$. By Proposition 2, $\alpha_1(p) = 1$ implies that $k_1 \leq \beta$. Thus, $k_1 < \tilde{\beta}$, which by Proposition 1 implies that $\tilde{\alpha}_1(p) = 1$. Next, consider the case where $n > 1$, and assume $\alpha_1(p) = 1$. By Proposition 1, this implies that $k_n < \beta$ and $p \leq p_n^* \equiv \frac{\beta - k_n}{\beta - k_n/n}$. Further note that

$$\tilde{p}_n^* \equiv \frac{\tilde{\beta} - k_n}{\tilde{\beta} - k_n/n} > \frac{\beta - k_n}{\beta - k_n/n} \equiv p_n^*.$$

Thus, $k_1 < \tilde{\beta}$ and $p < \tilde{p}_n^*$, which by Proposition 1 implies $\tilde{\alpha}_n(p) = 1$.

Now, suppose that $\alpha_n(p) < 1$. We wish to show that $\tilde{\alpha}_n(p) > \alpha_n(p)$. Suppose by contradiction that $\tilde{\alpha}_n(p) \leq \alpha_n(p)$. Now, it follows from Proposition 2 that if $k_n > \tilde{\beta}$,

$$\lim_{q \to 0+} \alpha_n(q) = \beta / k_n < \tilde{\beta} / k_n = \lim_{q \to 0+} \tilde{\alpha}_n(q).$$

Meanwhile, if $k_n \leq \tilde{\beta}$.

$$\lim_{q \to \tilde{p}_n^* +} \alpha_n(q) < 1 = \lim_{q \to \tilde{p}_n^* +} \tilde{\alpha}_n(q)$$

To see why the latter must must hold, first consider the case where $n = 1$. It follows from Lemma 5 that $\tilde{\alpha}_n(q) = 1$ for all q. Meanwhile, it follows again from Proposition 2 that $\alpha_1(q)$ is constant in q, and because $\alpha_1(p) < 1$, $\lim_{q \to \tilde{p}_n^* +} \alpha_n(q) < 1$. In the case where $n = 2$, because $p_n^* < \tilde{p}_n^*$, it follows from Proposition 1 that $\alpha_n(\tilde{p}_n^*) < 1$.

Thus, we have that both when $k_n > \tilde{\beta}$ and when $k_n \leq \tilde{\beta}$, there exists some $\hat{p} < p$ such that $\tilde{\alpha}_n(\hat{p}) > \alpha_n(\hat{p})$ and $\tilde{\alpha}_n$, α_n satisfy (ODE) on $[\hat{p}, p]$, for their respective value of β. Thus, there exists a $q \in [\hat{p}, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha'_n(q) \geq \tilde{\alpha}'_n(q)$. It follows from (ODE) that in order for the above two conditions to hold, it must be that

$$X \equiv (\beta - \tilde{\beta})(\frac{1-\alpha_n(q)}{\alpha_n(q)})(1-q) + \frac{V(q^i, n+1) - \tilde{V}(q^i, n+1)}{\alpha_n(q)} \geq 0 \tag{28}$$

where V and $\tilde{V}$ denote the value functions under β and $\tilde{\beta}$, respectively. First consider the case where $n = 1$. Then $V(q^i, n+1) = V(\tilde{q}^i, n+1) = 0$, and thus $X < 0$, contradicting (28).

Next, consider the case where $n > 1$. First suppose that $\alpha_{n+1}(q^i) = 1$. It follows from the inductive assumption that $\tilde{\alpha}_{n+1}(q^i) = 1$. Thus, by Lemma 5, $V(q^i, n+1) = \frac{k_{n+1}q^i}{N-n} = \tilde{V}(q^i, n+1)$. Again this implies that $X < 0$, contradicting (28). Now, suppose that $\alpha_{n+1}(q^i) < 1$. It then follows from Lemma 3 that

$$V(q^i, n+1) = V(q^i, n+1, \delta_0) = k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i)$$

Furthermore,

$$\tilde{V}(q^i, n+1) = \tilde{V}(q^i, n+1, \delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \tilde{\beta}(1-q^i)$$

Thus, recalling from (5) that $q_i = \alpha_{n+1}(q) + (1-\alpha_{n+1}(q))q$, we have

$$V(q^i, n+1) - \tilde{V}(q^i, n+1) \leq k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i))$$

Substituting this into the above expression for X, we obtain

$$X \leq \frac{k_{n+1}(\alpha_{n+1}(q^i) - \tilde{\alpha}_{n+1}(q^i))}{\alpha_n(q)} < 0.$$

where the strict inequality follows from the inductive assumption that $\alpha_{n+1}(q^i) < \tilde{\alpha}_{n+1}(q^i))$. Again, this is a contradiction of (28).

Next, let us establish part (b). Let $\tilde{\lambda} > \lambda > 0$, and let α, $\tilde{\alpha}$ denote the equilibria under λ and $\tilde{\lambda}$, respectively, fixing all other parameters. We begin by showing that $\tilde{\alpha}_n(p) = \tilde{\alpha}_n(p)$ for any p and n. Fix an n and assume inductively that if $n > 1$, $\alpha_{n+1}(p) = \tilde{\alpha}_{n+1}(p)$ for all p on-path.

Letting V, $\tilde{V}$ denote the value functions under the equilibria associated with λ and $\tilde{\lambda}$, respectively. Note that $V(p, n+1) = \tilde{V}(p, n+1)$ for all p on-path. In the case where $n = 1$, $V(p, n+1) = \tilde{V}(p, n+1) = 0$, and thus this holds trivially. In the case where $n > 1$, this follows from the inductive assumption.

Now, note that by Lemma 5, α_n and $\tilde{\alpha}_n$ must both be a solution to (P) at all (p, n) on-path, which does not depend on λ. By Theorem 1, the solution to (P) is unique, and $\alpha_n(p) = \tilde{\alpha}_n(p)$ at all (p, n) on-path.

Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under λ and $\tilde{\lambda}$, respectively. It then follows from (4) that $p(t) > \tilde{p}(t)$ for all $t > 0$. Thus, because $\alpha(p)$ and $\tilde{\alpha}(p)$ are both weakly decreasing in p (Proposition 3), it follows that $\alpha_n(p(t)) \leq \tilde{\alpha}_n(p(t))$. Furthermore, since $\tilde{\alpha}(p)$ is strictly decreasing in p (Proposition 3) whenever $\alpha_n(p) < 1$ and $k_N > \beta$, it follows that $\alpha_n(p(t)) < \alpha_n(\tilde{p(t)})$ in this case.

Finally, let us establish part (c). Let α and $\tilde{\alpha}$ denote the equilibria under N and $N + 1$ firms, respectively, fixing all other parameters. We begin by showing that for all p, $\alpha_n(p) \geq \tilde{\alpha}_n(p)$, and $\alpha_n(p) > \tilde{\alpha}_n(p)$ when $alpha_n(p) < 1$. To this end, fix an $n \in \{1, ..., N\}$ and assume inductively that the claim holds for $n + 1$ whenever $n < N$.

We begin by showing that $\tilde{\alpha}_n(p) = 1$ implies that $\alpha_n(p) = 1$. Suppose that $\tilde{\alpha}_n(p) = 1$. By Proposition 1, $\beta > k_n$ and $p < \tilde{p} + n^* \equiv \frac{\beta - k_n}{\beta - k_n/(N+1-n)}$. Note that because $p^*_n \equiv \frac{\beta - k_n}{\beta - k_n/(N-n)} > \tilde{p}^*_n$, it follows from Proposition 1 that $\alpha_n(p) = 1$.

Now consider the case where $\tilde{\alpha}_n(p) < 1$. We wish tot show that $\tilde{\alpha}_n(p) < \alpha_n(p)$. To this end, we begin by making a useful observation:

$$\text{If } \alpha_n \text{ and } \tilde{\alpha}_n \text{ both satisfy (ODE) at } q, \text{ and } \alpha_n(q) = \tilde{\alpha}_n(q), \text{ then } \alpha'_n(q) > \tilde{\alpha}'_n(q). \tag{29}$$

We will now establish this. Note first that for α_n and $\tilde{\alpha}_n$ to both satisfy (ODE) at q, given that $\alpha_n(q) = \tilde{\alpha}_n(q)$, the following must hold:

$$\alpha'_n(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n}{N-n+1} (k_n \alpha_n(q) - V(q^i, n+1) - \beta(1 - \alpha_n(q))(1-q))$$

$$\tilde{\alpha}'_n(q) = \frac{-1}{k_n(1-q)\alpha_n(q)} \frac{N-n+1}{N-n+2} (k_n \alpha_n(q) - \tilde{V}(q^i, n+1) - \beta(1 - \alpha_n(q))(1-q)),$$

where V and $\tilde{V}$ denote the value functions under the equilibria with N and $N+1$ total firms, respectively. Note that if $n = N$, $\alpha'_n(q) = 0$. Meanwhile, by Proposition 3, $\tilde{\alpha}'_n(q) < 0$. Thus, $\tilde{\alpha}'_n(q) < \alpha_n(q)$ must hold. Next, consider the case where $n < N$. We begin by observting that $V(q^i, n+1) > \tilde{V}(q^i, n+1)$. To see why this must hold, first consider the case where $\tilde{\alpha}_{n+1}(q^i) = 1$. It then follows from the inductive assumption that $\alpha_n(q^i) = 1$. Then, by Lemma 5,

$$\tilde{V}(q^i, n+1) = \tilde{V}(q^i, n+1, \delta_\infty) = \frac{k_{n+1} q^i}{N-n} < \frac{k_{n+1} q^i}{N-n-1} = V(q^i, n+1, \delta_\infty) = V(q^i, n+1).$$

Next, consider the case where $\tilde{\alpha}_n(q^i) < 1$. In this case, tif ollows from Lemma 3 that

$$\tilde{V}(q^i, n+1) = \tilde{V}(q^i, n+1, \delta_0) = k_{n+1}\tilde{\alpha}_{n+1}(q^i) - \beta(1-q^i) < k_{n+1}\alpha_{n+1}(q^i) - \beta(1-q^i)$$
$$= V(q^i, n+1, \delta_0) \leq V(q^i, n+1)$$

where the strict inequality follows from the inductive assumption made above. Examining the two ODEs listed above, since by Proposition 3, $\alpha'_n(q) \leq 0$, it follows that $\tilde{\alpha}'_n(q) < \alpha'_n(q)$.

Now, assume by contradiction that $\alpha_n(p) \leq \tilde{\alpha}_n(p)$. We begin by showing that there exists a $q^* < p$ such that $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$. To show this, first consider the case where $k_n \geq \beta$. Then, by Proposition 2,

$$\lim_{q \to 0+} \alpha_n(q) = \lim_{q \to 0+} \tilde{\alpha}_n(q) = \frac{\beta}{k_n}$$

Then, by the continuous differentiability of α_n and $\tilde{\alpha}_n$ on $(0, p)$, it follows from Equation 29 that for some $q^* < p$ sufficiently small $\alpha_n(q^*) > \tilde{\alpha}_n(q^*)$. Next, consider the case where $k_n < \beta$, and let $p_n^* \equiv \frac{\beta - k_n}{\beta/(N-n+1) - k_n}$. Note by Proposition 1 that $\alpha_n(p_n^*) = 1$. Meanwhile, because $p_n^* < \tilde{p}_n^* \equiv \frac{\beta - k_n}{\beta/(N-n+2) - k_n}$, it follows from Proposition 1 that $\tilde{\alpha}_n(p_n^*) < 1$, and thus, we have for $q^* = p_n^*$, $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$.

Since $\tilde{\alpha}_n(q^*) < \alpha_n(q^*)$ and $\tilde{\alpha}_n(p) \geq \alpha_n(p)$, by the continuous differentiability of α on $[q^*, p]$, there must exist some $q \in (q^*, p]$ such that $\alpha_n(q) = \tilde{\alpha}_n(q)$ and $\alpha'_n(q) \leq \tilde{\alpha}'_n(q)$. However, this is a contradiction of (29).

Now fixing any p and n, let $p(t)$ and $\tilde{p}(t)$ denote the common beliefs under N and $N+1$ firms, respectively. We wish to show that on some interval $[0, \bar{t}]$, where $\bar{t} > 0$, $\alpha_n(p(t)) \geq \tilde{\alpha}_n(\tilde{p}(t))$ is weakly increasing in t, and strictly so whenever $\alpha_n(p(t)) < 1$. First consider the case where $\alpha_n(p(t)) = 1$. In this case, the statement holds trivially. Next, consider the case where $\alpha_n(p) < 1$. It follows from the above that $\alpha_n(p) > \tilde{\alpha}_n(p)$. Now note that it follows from (4) that $\lim_{t \to 0+} p(t) - \tilde{p}(t) = 0$. Since $\alpha_n(p(t))$ and $\tilde{\alpha}_n(\tilde{p}(t))$ are both continuous in t (Lemma 4), it follows that for some $\bar{t} > 0$, $\alpha_n(p(t)) > \tilde{\alpha}_n(\tilde{p}(t))$ for all $t \in [0, \bar{t}]$. □

Appendix H Heterogeneous learning abilities: proofs

In the analysis that follows, we will take as given that Lemma 1-4 apply to each firm i's strategy and credibility function under the general model in which firms possess heterogeneous learning abilities. Formal proofs of this are omitted as the above proofs under the baseline model hold analogously under the extended model.

We begin by establishing an extension of Proposition 1 to the extended model. This claim is presented as Proposition 1′ below. In the analysis below, we let V_n^i denote the value function associated with firm i when they are the nth firm to report.

Proposition 1′. *For all s, there exists a $p_S^{i*} \in (0, 1]$ such that at any p on-path, $\alpha_1^i(p) = 1$ if and only if the following two conditions hold:*

1. $k_1 \leq \beta$
2. $p \leq p^{i*}$

Furthermore, $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$ and $n > 1$.

Proof. Fix an i. Suppose that $k_1 \leq \beta$. By identical reasoning as Proposition 1, for all $q < \frac{\beta - k_1}{k_1}$, $\alpha_1^i(q) = 1$. Let

$$p^{i*} \equiv sup\{p|\alpha_1^i(p) = 1 \text{ for all } q < p\}$$

Note that it follows by definition that $\alpha_1^i(p) = 1$ for all $p \leq p_n^{i*}$.

Next, we will show that $\alpha_1^i(q) < 1$ whenever $k_1 > \beta$ or $p > p_1^{i*}$. Suppose not by contradiction. First, consider the case where $k_1 > \beta$ and $\alpha_1^i(p) = 1$ for some p. Then we have that

$$V_1^i(p, \delta_0) = k_1 p + (k_1 - \beta)(1 - p) > k_1 p \leq V_1^i(p, \delta_\infty)$$

Thus, i can profitably deviate at p. Contradiction. Next, consider the case where $q > p_n^{i*}$ and $\alpha_1^i(p) = 1$. Here, a contradiction follows from identical reasoning to what is presented in Proposition 1.

Finally, we show that $p^{j*} > p^{i*}$ whenever $\lambda^j > \lambda^i$. Suppose by contradiction that $p^{j*} \leq p^{i*}$. Note that because j is truth telling at $(n = 1, p_S^{j*})$, $V_1^j(p_S^{j*}, \delta_\infty) \geq V_1^j(p^{j*}, \delta_0)$. Furthermore, because $p^{j*} \leq p^{i*}$, i is also truthful at $(n = 1, p_n^{j^*})$. Thus,

$$V_1^j(p_S^{j*}, \delta_0) = V_1^i(p_S^{j*}, \delta_\infty) = k_1 - \beta(1 - p).$$

Now, note that because $\lambda^j > \lambda^i$,

$$V_1^j(p_S^{j*}, \delta_\infty) > V_1^i(p_S^{j*}, \delta_\infty).$$

Combining these inequalities we have $V_1^i(p_S^{j*}, \delta_\infty) < V_1^i(p_S^{j*}, \delta_0)$. However, because $\alpha_1^i(p^{j*}) = 1$, $V_1^j(p_n^{j*}) = V_1^j(p_n^{j*}, \delta_\infty)$. Contradiction. □

Next, we extend Proposition 2 to this setting. Note this entails deriving an ODE which applies specifically to this setting, (ODE′).

Proposition 2′. *In equilibrium, for any p on-path, if $k_1 \geq \beta$ or $p > p^{i*}$, then the following ODE must be satisfied:*

$$\alpha_1^{i\prime}(p) = -\beta - \frac{\sum_{j \neq i} \frac{\lambda^j}{\alpha_1^j(p)}}{\sum_j \lambda^j (1-p)} [\alpha^i(p) - \beta(1-p)] \tag{ODE′}$$

In addition, $\lim_{p \to 0+} \alpha_1^i(p) = \beta/k_1$ must hold if $k_1 > \beta$, and $\lim_{p \to p^{i*}+} \alpha_1^i(p) = 1$ if $k_1 \leq \beta$.

Proof. Let us first establish that Equation ODE′ must hold under the conditions specified.

When $k_1 \geq \beta$ or $p > p^{i*}$, it follows from Proposition 1′ that $\alpha_1^i(p(t)) < 1$. It then follows from Lemma 3 that there exsits an $\varepsilon > 0$ such that for all $\Delta \in (0, \varepsilon)$,

$$\frac{V_{p,1}^i(\delta_\Delta) - V_{p,1}^i(\delta_0)}{\Delta} = 0$$

Recall that $V_{p,1}^i(\delta_0) = k_1 \alpha_1^i(p) - \beta(1-p)$. Meanwhile,

$$V_{p,1}^i(\delta_\Delta) = \int_0^\Delta k_1 \alpha_1^i(p(s)) \Psi(i,s) ds + (1 - \sum_j \lim_{s \to \Delta-} \Psi(j,s))[k_1 \alpha_1(p(\Delta)) - \beta(1 - p(\Delta))]$$

where Ψ is the first-report distribution associated with the strategy profile in which i plays δ_∞ and all $j \neq i$ play $F_{p,1}$. Specifically, for all $s > 0$,

$$\Psi(i,s) = p\lambda^i \int_0^s e^{-\prod_{j \in S} \lambda^j r} \prod_{j \neq i} (1 - F_{p,1}^i(r)) dr$$

and for $j \neq i$,

$$\Psi(j,s) = p \int_0^s e^{-\prod_{k \neq j} \lambda^k r} \prod_{k \neq i \neq j} (1 - F_{p,1}^k(r)) d(-e^{-\lambda^j r}(1 - F_{p,1}^j(r)))$$
$$+ (1-p) \int_0^s \prod_{k \neq k \neq j} (1 - F_{p,1}^k(r)) dF_{p,1}^j(r)$$

Substituting these two expressions into the above equation for $V_{p,1}^i(\delta_0)$ and following the same sequence of steps in Proposition 2 yields (ODE′).

Finally, the two limit conditions are established by the same reasoning presented in Proposition 2. □

Proof of Proposition 4. First suppose $\alpha_1^i(p) = 1$. It trivially holds that $\alpha_1^i(p) \geq \alpha_1^j(p)$ in this case.

Next, suppose $\alpha_1^i(p) < 1$. We want to show that $\alpha_1^i(p) > \alpha_1^i(p)$. Suppose by contradiction that $\alpha_1^i(p) \leq \alpha_1^j(p)$. Furthermore, first consider the case where $k_1 < \beta$. Then, let

$$q^* \equiv \inf\{q|\alpha_1^j(p) < 1 \text{ and } \alpha_1^j(p) < \alpha_1^i(p)\}.$$

Because the α_1^i are continuous, it follows from Proposition 1′, and the assumption that $\alpha_1^i(p) \leq \alpha_1^j(p)$, that $q^* < p$ exists. Again, by continuity, $\alpha_1^j(q^*) = \alpha_1^i(q^*)$. It then follows from (ODE′) that $\alpha_1^{j\prime}(q^*) < \alpha_1^{i\prime}(q^*)$. But this implies that for some $q > q^*$, $\alpha_1^j(q^*) > \alpha_1^i(q^*)$. Contradiction.

Next, consider the case where $k_1 \geq \beta$. Recall by Proposition 2′ that $\lim_{p\to 0+} \alpha_1^i(p) = \lim_{p\to 0+} \alpha_1^j(p)$. Thus, there exists some $q \in (0, p]$ such that $\alpha_1^i(p) \leq \alpha_1^j(p)$ and $\alpha_1^{i\prime}(p) \leq \alpha_1^{j\prime}(p)$. However, it again follows from (ODE′) that $\alpha_1^{i\prime}(p) > \alpha_1^{j\prime}(p)$. Contradiction.

□