Stable Assignments and Search Frictions

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1. Introduction

What we do

- We embed a static assignment problem with transferable utility into a dynamic search model . . .
  - in such a way that we can identify steady-state equilibrium outcomes of the dynamic model with feasible outcomes of the static assignment problem.
- We investigate the limits of steady-state equilibrium outcomes as the velocity of the search technology goes to infinity . . .
  - and ask whether such limit outcomes correspond to stable outcomes of the underlying static assignment problem.
1. Introduction
Why we do it

Investigate intuition that stable assignments are a shortcut to model situations in which frictions are negligible.

What has been done before

- Convergence to competitive equilibria in dynamic matching and bargaining games:
  Gale (JET 1987), Rubinstein and Wolinsky (Econometrica 1985), Lauermann (AER 2013), Cho and Matsui (JET 2017)

- Convergence to stable matchings in the marriage problem (NTU):
  Adachi (JET 2003), Lauermann and Nöldeke (JET 2014)
2. Assignment Problem

Assignment problem given by \((B, S, v, f)\):

- \(B\) and \(S\): disjoint, non-empty and finite set of agent types (buyers and sellers)
  
  - \(T = B \cup S\)

- \(v(b, s)\): value of a match between a buyer of type \(b\) and a seller of type \(s\)
  
  - Value of staying single/unmatched is normalized to zero
  - Transferable utility

- \(f(t) > 0\): mass of agents with type \(t \in T\)
2. Assignment Problem

- **Feasible assignment:** $x: B \times S \to \mathbb{R}$ satisfying

  
  $x(b, s) \geq 0$ for all $(b, s) \in B \times S$
  
  $x(t, t) \geq 0$ for all $t \in T$

  where

  
  $x(b, b) = f(b) - \sum_{s \in S} x(b, s)$
  
  $x(s, s) = f(s) - \sum_{b \in B} x(b, s)$

- **Optimal assignment** solves

  
  $$
  \max_{x \text{ feasible}} V(x) = \sum_{b \in B} \sum_{s \in S} x(b, s) v(b, s)
  $$
2. Assignment Problem

- Feasible outcome \((x, u)\): a feasible assignment \(x\) together with a payoff profile \(u : T \to \mathbb{R}\) satisfying

\[
\sum_{t \in T} f(t)u(t) = V(x)
\]

- A feasible outcome is individually rational if

\[u(t) \geq 0, \text{ for all } t \in T\]

and pairwise stable if

\[u(b) + u(s) \geq v(b, s) \text{ for all } (b, s) \in B \times S\]

- A feasible outcome is stable if it is individually rational and pairwise stable
2. Assignment Problem

Recall basic results:

1. Optimal assignments and stable outcomes exist
2. If \((x, u)\) is stable, then \(x\) is optimal

Assumption 1

- **There exists** \((b, s)\) such that \(v(b, s) > 0\)
- \(v(b, s) > 0 \Rightarrow v(b, s') \neq v(b, s)\) and \(v(b', s) \neq v(b, s)\) for all \(b \neq b'\) and \(s \neq s'\)
3. Search
Framework

- Random-search model in continuous time
- Mass \( f(t) > 0 \) of agents of each type \( t \) are “born” and enter the market per unit time
- Market is in steady-state with mass \( F(t) > 0 \) of agents of type \( t \) searching for a partner
- At rate \( \delta > 0 \) agents are exogenously removed from the market and become single with payoff of zero
- Meetings between agents are generated by a quadratic search technology with velocity parameter \( \lambda > 0 \):

\[
\lambda F(b)F(s) > 0
\]

is the mass of agents of type \( b \) that meet agents of type \( s \) per unit time
3. Search
Framework

- When two agents meet:
  - they observe each other’s type
  - each agent is selected with probability 0.5 to make a proposal for the division of $v(b, s)$; the other agent accepts or rejects
  - if the proposal is accepted, both agents leave the market and receive their agreed shares of $v(b, s)$
  - if the proposal is rejected, both agents continue to search

- Agents are risk neutral and there is no (further) discounting

- Remarks:
  - Framework as in Shimer and Smith (Econometrica 2000)
  - Quadratic search technology is an innocent simplification (Lauermann, Nöldeke, Tröger, Econometrica 2020)
3. Search
Steady state equilibrium

- Let $\alpha : B \times S \rightarrow [0, 1]$ specify the (stationary) fractions $\alpha(b, s)$ of meetings between agents with types $b$ and $s$ that result in a match.
- Payoff profile $u$ specifies the expected payoffs of those agents who are currently searching for a partner.
- A (steady-state) equilibrium is a triple $(\alpha, F, u)$ satisfying:
  1. Inflows and outflows balance for all types.
  2. For the given $u$, the specification of $\alpha$ is consistent with (subgame perfect) equilibrium in the induced bargaining games.
  3. Expected payoffs solve the appropriate value equations.
3. Search

Definition 1 (Equilibrium)

\((\alpha, F, u)\) is an equilibrium if for all \(b\) and \(s\):

\[
\begin{align*}
\delta u(b) &= \sum_{s \in S} \lambda F(s) \max\{0, v(b, s) - u(b) - u(s)\} / 2 \\
\delta u(s) &= \sum_{b \in B} \lambda F(b) \max\{0, v(b, s) - u(b) - v(s)\} / 2
\end{align*}
\]
3. Search
Equilibrium Outcomes

- Every equilibrium \((\alpha, F, u)\) induces an equilibrium outcome \((x, u)\), which describes what happens to a “cohort” of agents entering the market.
- \(u\) is the solution to the equilibrium value conditions and

\[
x(b, s) = \lambda \cdot \alpha(b, s) \cdot F(b) \cdot F(s) \geq 0
\]

- Equilibrium outcomes
  - exist (Lauermann and Nöldeke, Economics Letters 2015)
  - are feasible for the assignment problem \((B, S, v, f)\)
  - are individually rational
  - are never pairwise stable
4. Limit outcomes

**Definition 2**
An outcome \((x^*, u^*)\) is a limit outcome if there exists a sequence 
\((\lambda_k, x_k, u_k)_{k=1}^{\infty}\) such that

- \(\lambda_k \to \infty\)
- \((x_k, u_k)\) is an equilibrium outcome for the search model with velocity parameter \(\lambda_k\)
- \((x_k, u_k) \to (x^*, u^*)\)

- Limit outcomes exist, are feasible and individually rational.
- The question is whether they are pairwise stable, too . . . .
We find:

1. Limit outcomes may fail to be stable.
2. If a limit outcome is unstable, then it must feature excessive matching – frictions cause too much trade.
3. Simple sufficient conditions ensuring the stability of all limit outcomes.
4. Bounds on the efficiency loss that may arise in a limit outcome.
4. Limit outcomes

Example with unstable limit outcome

- \( B = \{b_1, b_2\}, \ S = \{s_1, s_2\} \)
- \( f(b_1) = 9, \ f(b_2) = 2, \ f(s_1) = 9, \ f(s_2) = 1 \)
- \( v(b, s) \) given by

\[
\begin{array}{cc|cc}
 & s_2 & s_1 \\
 b_2 & 10 & 2 \\
b_1 & 2 & -6 \\
\end{array}
\]

- Unique stable outcome \((\hat{x}, \hat{u})\):

\[
\hat{x} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

and

\[
\hat{u}(b_1) = 0, \ \hat{u}(b_2) = 2, \ \hat{u}(s_1) = 0, \ \hat{u}(s_2) = 8
\]
4. Limit outcomes

Example with unstable limit outcome

- There is (another) limit outcome given by

\[ x^* = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \]

and

\[ u^*(b_1) = 0, \quad u^*(b_2) = 2, \quad u^*(s_1) = 0, \quad u^*(s_2) = 2 \]

- The unstable limit outcome is supported by a sequence of equilibria in which all matches with \( v(b,s) > 0 \) are consummated.

- For high \( \lambda \), these equilibria reflect a coordination failure: high-value agents on one side of the market are too eager to match because high-value agents on the other side of the market are also too eager too match.

- This example is robust.
4. Limit outcomes

Some Terminology

Let \((x^*, u^*)\) be a limit outcome. We say that

- Type \(t\) is fully matched if \(x^*(t, t) = 0\)
- Type \(t\) is partially matched if \(0 < x^*(t, t) < f(t)\)
- Type \(t\) is unmatched if \(x^*(t, t) = f(t)\)
- Type pair \((b, s)\) is a blocking pair if \(u^*(b) + u^*(s) < v(b, s)\)
4. Limit outcomes

Properties of limit outcomes

Lemma 1

Let \((x^*, u^*)\) be a limit outcome in which \((b, s)\) is a blocking pair. Then

1. \(b\) and \(s\) are fully matched: \(x^*(b, b) = x^*(s, s) = 0\)
2. \(b\) and \(s\) obtain strictly positive payoffs:

\[
    u^*(b) > 0 \quad u^*(s) > 0
\]

3. \(b\) and \(s\) do not match with any fully matched types:

\[
    x^*(s', s') = 0 \Rightarrow x^*(b, s') = 0, \quad x^*(b', b') = 0 \Rightarrow x^*(b', s) = 0
\]

(In particular, they do not match with each other)
4. Limit outcomes

Properties of limit outcomes

Lemma 2

Let \((x^*, u^*)\) be a limit outcome. Then

1. Types that are not fully matched receive a payoff of zero:

   \[ x^*(t, t) > 0 \implies u^*(t) = 0 \]

2. Types that match with each other share the value of the corresponding match:

   \[ x^*(b, s) > 0 \implies u^*(b) + u^*(s) = v(b, s) \]
4. Limit outcomes

Properties of limit outcomes

Lemma 3

Let \((x^*, u^*)\) be a limit outcome in which \((b, s)\) is a blocking pair. Then there exist partly matched types \(b' \neq b\) and \(s' \neq s\) such that \(b\) and \(s\) are fully matched with these types:

\[
\begin{align*}
x^*(b, s') &= f(b) \\
x^*(b', s) &= f(s)
\end{align*}
\]

Proof (for \(b\); same argument applies to \(s\)):

- \(b\) obtains a strictly positive payoff and must be fully matched (Lemma 1)
- \(\ldots\) with types that are partially matched (Lemma 1)
- Partners of \(b\) obtain payoff of zero (Lemma 2). Hence, all \(b\)-agents match with the same partner type, \(s'\) (genericity assumption on \(v\))
4. Limit outcomes

Properties of limit outcomes

Proposition 1

Let \((x^*, u^*)\) be an unstable limit outcome. Then there exist a stable outcome \((\hat{x}, \hat{u})\) such that

\[
\hat{x}(t, t) \geq x^*(t, t) \\
\hat{u}(t) \geq u^*(t)
\]

holds for all types \(t\) and

\[
\sum_t \hat{x}(t, t) > \sum_t x^*(t, t) \\
V(\hat{x}) > V(x^*)
\]

Proof exploits Lemma 3 to construct an auxiliary assignment problem involving only types in blocking pairs.
4. Limit outcomes
Sufficient conditions for stability

Proposition 2
Every limit outcome is stable if all types have the same mass.

Proposition 3
Every limit outcome is stable if $v(b, s) > 0$ holds for all types.
4. Limit outcomes
Bounding the efficiency loss

Assumption 2 (Monotonicity)

The sets \( B \) and \( S \) are totally ordered and

\[
v(b, s) > 0 \Rightarrow v(b', s) > v(b, s) \quad \text{and} \quad v(b, s') > v(b, s)
\]

for all \( b' > b \) and \( s' > s \).

Monotonicity ensures that all blocking types must match with the same type on the other side of the market.
4. Limit outcomes
Bounding the efficiency loss

Define

- $\bar{f} = \max_t f(t)$
- $\bar{v} = \max_{(b,s)} v(b,s)$

**Proposition 4**

Suppose the assignment problem is monotonic. Let $x^*$ be a limit assignment and $\hat{x}$ be an optimal assignment. Then

$$V(x^*) \geq V(\hat{x}) - 2\bar{f}\bar{v}$$

Not the best possible bound – it only exploits the structure identified in Lemma 3