Matching in Dynamic Imbalanced Markets

Itai Ashlagi    Afshin Nikzad    Philipp Strack
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1. Matching faster reduces waiting time (often costly, i.e. time on dialysis)
2. Waiting “thickens” the market and might facilitate more and better matches.
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→ We look at this question in large kidney exchanges without match quality.
Kidney Exchanges

- About 18,000 transplant per year:
  - 12,000 from cadaver organs;
  - 6,000 from living donors;
  - 5,000 are removed without a transplant (“too sick”) while 35,000 joined the list.

Live donation helps to overcome shortage:
- Direct donations: patient and compatible donor.
- Exchange: between incompatible patient-donor pairs (≈15% of US live donor transplants).

Exchanges match at different speeds:
- Europe (Netherlands, UK, Czech Republic), Canada, Australia: 3-4 months.
- Israel: daily.

Concerns that high matching frequency in the US leads to inefficiency: “There has been a race to the bottom in that registries forced by competition to perform match runs very frequently...and likely fewer transplants are accomplished nationwide” (Gentry and Segev, AJT 2015).
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Related literature


• **Maximize number of matches with departures:** Akbarpour, Li, Oveis-Gharan 2020, Nikzad, Akbarpour, Rees, Roth 2020

• **Simulations:** Ashlagi et al. (AJT) 2017, Agarwal, Ashlagi, Azevedo, Featherstone, Karaduman 2018.


• **Dynamic programming for KE:** Dickerson, Procaccia, Sandholm 2012a,b
Overview
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- Analysis of the compatibility graph.

  1. data: Identify empirical features of large compatibility graphs.
  2. theory: Single type random graph models can not fit these features.
  3. theory: Propose a simple two-type model.

Analysis of dynamic matching policies.

  1. Consider three policies: (1) Greedy matching (2) Batching (3) Patient matching.
  2. Theoretical analysis:
      - Greedy matching is (almost) optimal in large markets for all types and all linear EU preferences.
      - Batching policies can only be optimal if they match at high frequency.
      - Patient is suboptimal in large markets.
  3. Empirical analysis:
      - Greedy does better than 7d, 14d, 30d batching for moderate market sizes both in simulations of the model and with real compatibility data.
      - Patient matches around 1% more, but leads to 35% longer waiting time.
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The compatibility graph
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- **Size of the maximum matching:**
  \[
  \text{SMM} = \max_{\mu \in M(G)} \frac{|\mu|}{|G|}.
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• Capture the benefit of enlarging the market exogenously or by waiting.
Figure 1: Average percentage of pairs without a compatible partner (dashed) and the percentage matched in a maximum matching (solid). The average for every fixed pool size on the horizontal axis is computed by random sampling from the combined data set from NKR, APD, UNOS and Methodist at San Antonio.
Empirical Regularities

Two empirical regularities:

1. Size of the maximum matching is bounded away from 1.
2. Fraction of agents without a partner goes to 0.
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Proposition

Consider a model with \( m \) homogeneous agents, in which every pair of agents are compatible independently with probability \( p(m) > 0 \) that may depend on the market size. The following two conditions cannot be satisfied simultaneously

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\lim_{m \to \infty} E[SMM] < 1, \quad \text{and} \quad \lim_{m \to \infty} E[FWP] = 0.
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$$\lim_{m \to \infty} \mathbb{E} [\text{FWP}] = 0.$$  

Intuitively, heterogeneity plays a major role.
A simple two-type model

Figure 2: The random compatibility model.

- Two types easy (E) and hard-to-match (H).
- Compatibility is independent across pairs.
  1. $p > 0$ between (E) and (H);
  2. $q > 0$ between (E) and (E);
  3. 0 between (H) and (H);
A simple two-type model

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Figure 2: The random compatibility model.
Proposition

Consider the compatibility model with $m$ easy-to-match agents and $(1 + \lambda)m$ hard-to-match agents where $\lambda > 0$. Then, with high probability\(^1\) we have that

\[
\text{SMM} = \frac{2}{2 + \lambda},
\]

(3)

\[
\text{FWP} = 0.
\]

(4)

\(^1\)A sequence of events $E_1, E_2, \ldots$ holds with high probability if there is $\alpha > 0$ such that

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\lim_{n \to \infty} n^\alpha (1 - P[E_n]) = 0.
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- There are \( \lambda m \) more H agents which go unmatched \( \Rightarrow SMM \leq \frac{2}{2 + \lambda} \).

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- Perfect matching in bipartite with high prob. \( \Rightarrow \text{SMM} = \frac{2}{2 + \lambda} \) with high prob.

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Relating this model to data

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- Perfect matching in bipartite with high prob. \( \Rightarrow \) \( SMM = \frac{2}{2 + \lambda} \) with high prob.
- In the data \( \lim_{m \to \infty} SMM = 0.6 \) suggesting \( \lambda \approx 1.3 \), i.e. 70\% hard-to-match.

\(^1\)A sequence of events \( E_1, E_2, \ldots \) holds with high probability if there is \( \alpha > 0 \) such that

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- Agents *arrive* according to Poisson process.

1. $H$ agents arrive at rate $m(1 + \lambda)$.
2. $E$ agents arrive at rate $m$.

- Agents become critical at an exponentially distributed time with mean $d$.
  1. Criticality is observable.
  2. Last time an agent can match.

- A dynamic policy chooses at each point in time a matching $\mu_t \in M(G_t)$ to execute.
  1. **Greedy**: execute every possible matching immediately.
  2. **Batching**: every $T$ units of time execute a maximal matching.
  3. **Patient**: whenever an agent gets critical attempt to match that agent.

Throughout break ties randomly in favor of $H$ agents.
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- A dynamic policy chooses at each point in time a matching $\mu_t \in M(G_t)$ to execute.
  1. **Greedy**: execute every possible matching immediately.
  2. **Batching**: every $T$ units of time execute a maximal matching.
  3. **Patient**: whenever an agent gets critical attempt to match that agent.

Throughout break ties randomly in favor of $H$ agents.
Dynamic Matching

- Agents **arrive** according to Poisson process.
  1. $H$ agents arrive at rate $m(1 + \lambda)$.
  2. $E$ agents arrive at rate $m$.

- Agents **become critical** at an exponentially distributed time with mean $d$.
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Throughout break ties randomly in favor of $H$ agents.
Why can waiting be beneficial?

Time 1:

\[ A_1 \rightarrow A_2 \]

• The policy which matches at time 1 matches 2 agents.
• The policy which matches at time 2 matches 4 agents.
Why can waiting be beneficial?

Time 1:

\[ A_1 - A_2 \]

Time 2:

\[ A_3 - A_1 - A_2 - A_4 \]
Why can waiting be beneficial?

Time 1:

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Time 2:

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Why can waiting be beneficial?

Time 1:

\[ A_1 \rightarrow A_2 \]

Time 2:

\[ A_3 \rightarrow A_1 \rightarrow A_2 \rightarrow A_4 \]

- The policy which matches at time 1 matches 2 agents.
- The policy which matches at time 2 matches 4 agents.
Measure for performance

- $\theta_i \in \{E, H\}$ agent $i$’s type
- $\alpha_i \geq 0$ her arrival time
- $\varphi_i \geq 0$ how long she is present in the market
- $\mu_i \in \{0, 1\}$ whether she is matched.
Measure for performance

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- $\alpha_i \geq 0$ her arrival time
- $\varphi_i \geq 0$ how long she is present in the market
- $\mu_i \in \{0, 1\}$ whether she is matched.

- **Match rate**

  $$q_{\Theta}(m) = \lim_{t \to \infty} \mathbb{E} \left[ \frac{|\{i: \mu_i = 1 \text{ and } \alpha_i \leq t \text{ and } \theta_i = \Theta\}|}{|\{i: \alpha_i \leq t \text{ and } \theta_i = \Theta\}|} \right].$$

- **Waiting time**

  $$w_{\Theta}(m) = \lim_{t \to \infty} \mathbb{E} \left[ \frac{\sum_{i: \alpha_i \leq t \text{ and } \theta_i = \Theta} \varphi_i}{|\{i: \alpha_i \leq t \text{ and } \theta_i = \Theta\}|} \right].$$

- Motivation: payoff of a risk-neutral expected-utility-maximizer with constant waiting cost.
**Definition (Asymptotic optimality)**
A policy is asymptotically optimal if for every $\epsilon > 0$ there exists $m_{\epsilon}$ such that, when $m \geq m_{\epsilon}$, no type of agent can improve its match rate $q_{\Theta}(m)$ or expected waiting time $w_{\Theta}(m)$ by more than $\epsilon$ when changing to any other policy.

**Demanding as the policy needs to be optimal**
1. for all types.
2. for all risk-neutral preferences with linear waiting cost.

**Existence of such a policy is unclear.**
If such a policy exists, there is no conflict between different types in a large market.
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A policy is asymptotically optimal if for every $\epsilon > 0$ there exists $m_\epsilon$ such that, when $m \geq m_\epsilon$, no type of agent can improve its match rate $q_\Theta(m)$ or expected waiting time $w_\Theta(m)$ by more than $\epsilon$ when changing to any other policy.

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- Existence of such a policy is unclear.
- If such a policy exists it there is no conflict between different types in a large market.
Results

Theorem

The greedy policy is asymptotically optimal, whereas the batching policy (for any fixed batch length) and the patient policy are not asymptotically optimal.
Theorem

The greedy policy is asymptotically optimal, whereas the batching policy (for any fixed batch length) and the patient policy are not asymptotically optimal.

- Greedy is (almost) optimal for H and E agents in sufficiently large markets.
- Patient and Batching with fixed batch length are suboptimal in sufficiently large markets.
**Proposition:** As the arrival rate $m$ grows large match rate and waiting time converge to:

<table>
<thead>
<tr>
<th></th>
<th>Matchrate $q$</th>
<th>Waiting time $w$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H$</td>
<td>$E$</td>
</tr>
<tr>
<td><strong>Greedy</strong></td>
<td>$\frac{1}{1+\lambda}$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Batching</strong></td>
<td>$\frac{1-e^{-T/d}}{(1+\lambda)T/d}$</td>
<td>$\frac{1-e^{-T/d}}{T/d}$</td>
</tr>
<tr>
<td><strong>Patient</strong></td>
<td>$\frac{1}{1+\lambda}$</td>
<td>1</td>
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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>Greedy</strong></td>
<td>$\frac{\lambda d}{1+\lambda}$</td>
<td>0</td>
</tr>
<tr>
<td><strong>Batching</strong></td>
<td>$d(1 - q_H)$</td>
<td>$d(1 - q_E)$</td>
</tr>
<tr>
<td><strong>Patient</strong></td>
<td>$d$</td>
<td>0</td>
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- $d$ expected criticality time
- $\lambda$ imbalance parameter
- $T$ batching time
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- $d$ expected criticality time
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- **Batching:**
  1. For $T \to 0$ Batching converges to Greedy (for a fixed market size).
  2. For $T > 0$ matches both fewer $(H, E)$ agents and matches the slower.
**Proposition:** As the arrival rate \( m \) grows large match rate and waiting time converge to:

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- \( d \) expected criticality time
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1. For $T \to 0$ Batching converges to Greedy (for a fixed market size).
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1. Achieves the optimal match rates.
2. Induces longer waiting times for $H$ agents, but not for $E$. 
Figure 3: Illustration when $\lambda = 1.33$ and $d$ equals 360 days. The blue points represent the predictions of our model for large markets which we derived.
• Match rates of \( \left( \frac{1}{1+\lambda}, 1 \right) \) are achieved by matching all \( E \) agents to \( H \) agents and thus constitute an upper bound.

• Clearly, a waiting time of 0 is a bound for the \( E \) waiting time.

• The upper bound on the waiting time for \( H \) agent is obtained by analyzing the process where all \( H \) and \( E \) agents are compatible.

• Intuitively, \( H \) agents must accumulate in any mechanism as they can only match to \( E \) agents and there are fewer \( E \) agents.

• As \( H \) agents accumulate \( E \) agents can always find a partner immediately.

• Under Greedy this achieves the upper bound in a large market.

• Under Batching the match rate is lower as some agents leave between matching intervals.

• Under Patient \( H \) agents get only matched when they get critical and thus wait a long time.

• Proofs: Detailed analysis of the 2-dimensional Markov chain for each process.
Proof Intuition

- Match rates of \((\frac{1}{1+\lambda}, 1)\) are achieved by matching all \(E\) agents to \(H\) agents and thus constitute an upper bound.
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Proof Intuition

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- Under Greedy this achieves the upper bound in a large market.
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- Under Batching the match rate is lower as some agents leave between matching intervals.
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- **Proofs:** Detailed analysis of the 2-dimensional Markov chain for each process.
How large do is large?
**Proposition:** A market size dependent batching policy with batch length $T_m$ is asymptotically optimal if and only if the batch length goes to zero as the market becomes large $\lim_{m \to \infty} T_m = 0$. 

For how large batching time is batching suboptimal at a fixed market size? 

**Proposition:** Let $m > 0$ be an arbitrary fixed arrival rate. Define $z^*$ to be the steady-state probability that an E agent, upon her arrival, is matched to an H agent under the greedy policy. Then, for every E and H agents the match rate and waiting time of that type under the batching policy are worse than under the greedy policy if $T > z^* W(\frac{-e^{-1}}{z^*}) + 1 \frac{z^*}{d(5)}$ and $W(\cdot)$ is the Lambert W function.
Proposition: A market size dependent batching policy with batch length $T_m$ is asymptotically optimal if and only if the batch length goes to zero as the market becomes large
\[ \lim_{m \to \infty} T_m = 0. \]

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\[
T > \frac{z^* W \left( -\frac{e^{-1/z^*}}{z^*} \right) + 1}{z^*/d} \tag{5}
\]

and \( W(\cdot) \) is the Lambert \( W \) function.
**Figure 4:** The batch length above which greedy dominates batching for various arrival rates per day, $\lambda = 1.33$, $p = 0.037$ and average criticality time $d = 360$ days. The bound $T^*$ is independent of $q \in [0, 1]$, and is decreasing in $p$. 
Figure 4: The batch length above which greedy dominates batching for various arrival rates per day, $\lambda = 1.33$, $p = 0.037$ and average criticality time $d = 360$ days. The bound $T^*$ is independent of $q \in [0, 1]$, and is decreasing in $p$.

For 1.6 pairs arriving per day matching batching less frequently than daily is strictly sub-optimal for all types (National Kidney Registry $\equiv$ 1 pair per day).
Model Simulations
(a) Different arrival rates

(b) Different imbalances

(c) Different compatibilities

(d) Match probabilities $p = 0.02, q = 1$
Data Simulations
Simulations

- We use compatibility data from 1881 de-identified patient-donor pairs from the NKR (between July 2007 to December 2014).
- Arrivals and departures follow our model ($d = 360$).
- We vary market size between $1/10$ and 4 times the size of the NKR.
- We simulate the arrival of 10 million pairs for each policy.
<table>
<thead>
<tr>
<th>arrivals per day</th>
<th>match rate</th>
<th>waiting time in days</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Greedy</td>
<td>Patient</td>
</tr>
<tr>
<td>0.01</td>
<td>10.7%</td>
<td>11.9%</td>
</tr>
<tr>
<td>0.05</td>
<td>22.4%</td>
<td>23.4%</td>
</tr>
<tr>
<td>0.25</td>
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<td>1</td>
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<td>48%</td>
</tr>
</tbody>
</table>

**Table 1:** Match rate and average waiting time over all pairs in simulations using NKR data.
<table>
<thead>
<tr>
<th>arrivals per day</th>
<th>Greedy match rate</th>
<th>Patient match rate</th>
<th>Batching match rate</th>
<th>waiting time in days</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10.7%</td>
<td>11.9%</td>
<td>10.4%</td>
<td>322</td>
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<tr>
<td>0.01</td>
<td>10.7%</td>
<td>11.9%</td>
<td>10.4%</td>
<td>322</td>
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<td>22.4%</td>
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**Table 1**: Match rate and average waiting time over all pairs in simulations using NKR data.

- Patient leads to 35% longer waiting times, but ≈ 1% higher match rate.
<table>
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<tr>
<th>arrivals per day</th>
<th>Greedy Patient</th>
<th>Batching 7 days 30 days 60 days</th>
<th>Greedy Patient</th>
<th>Batching 7 days 30 days 60 days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>10.7% 11.9%</td>
<td>10.4% 9.9% 9.3%</td>
<td>322 355</td>
<td>322 324 326</td>
</tr>
<tr>
<td>0.05</td>
<td>22.4% 23.4%</td>
<td>22.2% 21.2% 20.2%</td>
<td>279 324</td>
<td>280 283 288</td>
</tr>
<tr>
<td>0.25</td>
<td>34.3% 35.4%</td>
<td>33.8% 32.6% 31.2%</td>
<td>237 298</td>
<td>238 243 248</td>
</tr>
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**Table 1:** Match rate and average waiting time over all pairs in simulations using NKR data.

- Patient leads to 35% longer waiting times, but $\approx 1\%$ higher match rate.
- Greedy does better than $7d, 30d, 60d$ Batching in terms of waiting times and match rate.
Figure 6: Average waiting times (WT) and match rate (MR) in days under greedy (G) and patient (P) policies. The left and right axes are WT and MR. The label (*) excludes pairs who have no match in the data. Under-demanded patient-donor pairs are blood type incompatible, Over-demanded pairs are blood type compatible, but tissue type incompatible.
Figure 7: Averages of waiting times (left) and chance of matching (right) taken over copies for each pair in the data. The axes correspond to the greedy and patient policies. Arrival 1 per day.
Modelling Assumptions
Robustness

- Single Type.
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  • Simulations: results hold when there are not hard-to-match types \( \lambda = -1 \).
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  - Much simpler single type model cannot match the data and answer how hard-and-easy to match agents are differentially affected.
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  Akbarpour et al., (2020) show that the loss ratio between different policies can be infinitely better under patient matching.
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We looked at **when to match** in large kidney exchanges.

**Compatibility graph:**
- No single-type model can match aggregate features of the data.
- Simple and interpretable two type model that matches the data.

**Dynamic Matching:**
- Greedy matching is optimal in large markets.
  - For all risk-neutral EU preferences.
  - For hard and easy-to-match agents.
  - No trade-off between matching more agents and faster.

**Empirically at the size of the NKR.**
- Greedy outperforms weekly, monthly, bimonthly matching.
- Patient leads to $\geq 1\%$ higher match rate, but $35\%$ longer waiting time.
- Patient matching makes *easy-to-match* agents better off and hurts *hard-to-match* agents.
Thank You!
• “Small” markets:
  ● Merging will increase the match rate (Agarwal et al. 2018, 2019). Emerging collaborations between European countries.

• How to match with heterogeneous match qualities? (the next talk...)