Lottery Equilibrium

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Indivisibilities and non-convex preferences often present problems:

- general equilibrium theory
- market design

**Goal:** develop a unified and simple approach to these problems
Introduction

General equilibrium theory

- With indivisibilities and/or non-convex preferences:
  - competitive equilibria may fail to exist
  - competitive equilibria may be inefficient (in a sense)

- Simple solution:
  - allow traders to engage in (binary) lotteries
  - “lottery equilibrium”
Related literature

- **Lottery equilibrium in special cases:**
  - Rogerson (1988)
  - Hylland and Zeckhauser (1979); Budish, Che, Kojima and Milgrom (2013); Akbarpour and Nikzad (2017)

- **Competitive equilibrium from equal incomes:**
  - Varian (1974)
  - Budish (2011); Budish and Kessler (2016); Budish, Cachon, Kessler and Othman (2017)

- **Competitive equilibrium in continuum economies:**
  - Mas-Colell (1977)
Outline

Example

Model (Continuum Economy)

Results

- Existence
- First Welfare Theorem
- Second Welfare Theorem

Next Steps

- Finite Economy
- Market Design Applications
Example

Environment

- **Consumption set:** $\Omega = \mathbb{Z}_{\geq 0} \times [0, \infty) \times [0, \infty)$
  - one indivisible good “houses”
  - one divisible good “corn”
  - one divisible “artificial currency”

- **Binary lotteries:** $\Delta(\Omega)$

- **Agents:** $t \in T = [0, 1]$
  - utility function $u_t(a_t) = 3(1 + t)\mathbb{1}(a_t^1 \geq 1) + a_t^2$
  - endowment $\omega_t \in \Omega$

\[
\int \omega_t + \int \psi_t = \left( \frac{1}{2}, 1, 1 \right)
\]

“inside” endowment \hspace{1cm} “outside” endowment
Example
Competitive equilibrium vs. lottery equilibrium

Endowments (for now):

▶ no outside endowments: $∫ ψ_t = (0, 0, 0)$
▶ inside endowments: $ω_t ∈ \{(1, 0, 1), (0, 2, 1)\}$

Competitive equilibrium allocation:

▶ agents consume their endowments
▶ Pareto dominated (by a lottery allocation)
Example

Competitive equilibrium vs. lottery equilibrium

Endowments (for now):

- no outside endowments: \( \int \psi_t = (0, 0, 0) \)
- inside endowments: \( \omega_t \in \{(1, 0, 1), (0, 2, 1)\} \)

Lottery equilibrium allocation:

- \( t \leq \frac{1}{3} \) : \( a_t = \begin{cases} (0, 4, 1) & \text{if } \omega_t = (1, 0, 1) \\ (0, 2, 1) & \text{if } \omega_t = (0, 2, 1) \end{cases} \)
- \( t > \frac{1}{3} \) : \( a_t = \begin{cases} (1, 0, 1) & \text{if } \omega_t = (1, 0, 1) \\ \frac{1}{2} \cdot (1, 0, 1) + 0 \cdot (0, 0, 1) & \text{if } \omega_t = (0, 2, 1) \end{cases} \)

- Pareto efficient
Binary lotteries suffice

\(t = 1, \ p = \left( \frac{4}{5}, \frac{1}{5}, 0 \right)\)
Binary lotteries suffice

$t = 1, \ p = (\frac{4}{5}, \frac{1}{5}, 0)$

\[ \tilde{\nu}_t(p, w) \]
Binary lotteries suffice

\( t = 1, \ p = (\frac{4}{5}, \frac{1}{5}, 0), \ \omega_t = (0, 2, 1) \)
Binary lotteries suffice

\[ t = 1, \ p = \left( \frac{4}{5}, \frac{1}{5}, 0 \right), \ \omega_t = (0, 2, 1) \]
Example
Envy-freeness

An (ex ante) envy-free allocation:

\[ a_t = \frac{1}{2} \cdot (1, 1, 0) + \frac{1}{2} \cdot (0, 1, 0) \]
An (ex ante) envy-free allocation:
\[ \forall t : a_t = \frac{1}{2} \cdot (1, 1, 0) + \frac{1}{2} \cdot (0, 1, 0) \]

The efficient and envy-free allocation:
\[ t \leq \frac{1}{3} : a_t^* = (0, 3, 0) \]
\[ t > \frac{1}{3} : a_t^* = \frac{3}{4} \cdot (1, 0, 0) + \frac{1}{4} \cdot (0, 0, 0) \]
Example
Second Welfare Theorem

**Failure of 2WT:**
- suppose outside endowments are $\int \psi_t = (0, 0, 1)$
  and inside endowments satisfy $\int \omega_t = (\frac{1}{2}, 1, 0)$
- for all inside endowments $\omega : T \to \Omega$ and all price vectors $p$, $(p, a^*)$ is not a lottery equilibrium

**Success of 2WT:**
- suppose outside endowments are $\int \psi_t = (\frac{1}{2}, 1, 0)$
  and inside endowments are $\omega : t \mapsto (0, 0, 1)$
- $((\frac{1}{2}, \frac{1}{8}, \frac{3}{8}), a^*)$ is a lottery equilibrium
Example
Second Welfare Theorem

Failure of 2WT:
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- $((\frac{1}{2}, \frac{1}{8}, \frac{3}{8}), a^*)$ is a lottery equilibrium
- (in contrast, competitive equilibrium fails to exist)
Continuum model

Environment

- **Consumption set:** $\Omega = \mathbb{Z}_{\geq 0}^m \times [0, \infty)^n \times [0, \infty)$
  - $m$ indivisible goods
  - $n$ divisible goods
  - one divisible “artificial currency”

- **Binary lotteries:** $\Delta(\Omega)$

- **Agents:** $t \in T = [0, 1]$

- **Economy:** $e : T \rightarrow \mathcal{U} \times \Omega \times \Omega$
  - $u_t$: agent’s utility function (continuous, weakly increasing, constant in last component)
  - $\omega_t$: agent’s inside endowment
  - $\int \psi_t$: aggregate outside endowment
Lottery allocation: $a : T \rightarrow \Delta(\Omega)$ such that

$$\int \mathbb{E}[a_t] \leq \int \omega_t + \int \psi_t$$

- with equality in the first $m + n$ components
Continuum model
Lottery allocations

**Lottery allocation:** $a : T \to \Delta(\Omega)$ such that

$$\int E[a_t] \leq \int \omega_t + \int \psi_t$$

- with equality in the first $m + n$ components

**(Ex ante) Pareto efficiency:** there is no other lottery allocation $a'$ such that

- $u_t(a'_t) \geq u_t(a_t)$ for all $t \in T$
- $u_t(a'_t) > u_t(a_t)$ for all $t \in T' \subset T$, $\lambda(T') > 0$

**(Ex ante) envy-freeness:** $u_t(a_t) \geq u_t(a_s)$ for all $(s, t) \in T \times T$
Lottery equilibrium: \((p, a)\) where \(p \in \Delta\) and \(a\) is a lottery allocation such that for all \(t \in T\):

\[
\begin{align*}
\forall t \in T, a_t &\in B_t(p) := \{ a \in \Delta(\Omega) : p \cdot \mathbb{E}[a] \leq p \cdot \omega_t \} \\
\forall t \in T, a_t &\in C_t(p) := \arg \max_{a \in \Delta(\Omega) \cap B_t(p)} u_t(a) \\
\forall t \in T, a_t &\in D_t(p) := \arg \min_{a \in \Delta(\Omega) \cap C_t(p)} p \cdot \mathbb{E}[a]
\end{align*}
\]
Existence

**Theorem**

If an economy $e$ satisfies either Condition (A) or Condition (B), then there exists a lottery equilibrium $(p, a)$ for $e$.

**Condition (A)**

- each $u_t$ is strictly monotonic in the first $m + n$ components
- each $u_t$ is bounded above by a strictly concave function

**Condition (B)**

- each $u_t$ is satiated by some $\bar{a} \in \Omega$
- each $\omega_t^{m+n+1} > 0$
First Welfare Theorem

Theorem
If $(p, a)$ is a lottery equilibrium for $e$, then $a$ is Pareto efficient.
Second Welfare Theorem

**Theorem**

If

- \( e = (u, \omega, \psi) \) is an economy satisfying Condition (A)
- \( a^* \) is a Pareto efficient lottery allocation with \( a^*_t, m+n+1 = 0 \) for all \( t \)

Then there exists an economy \( \hat{e} = (\hat{u}, \hat{\omega}, \hat{\psi}) \) such that

- \( \hat{u} = u \) and \( \int \hat{\omega}_t + \int \hat{\psi}_t = \int \omega_t + \int \psi_t \)
- \( (p, a^*) \) is a lottery equilibrium for \( \hat{e} \) for some prices \( p \in \Delta \)
Lottery equilibrium from equal incomes (LEEI)

**Lemma**

If \((p, a)\) is a lottery equilibrium for an economy \(e\) in which \(\omega : T \rightarrow \Omega\) is a constant mapping, then \(a\) is envy-free.
Lottery equilibrium from equal incomes (LEEI)

Lemma
If \((p, a)\) is a lottery equilibrium for an economy \(e\) in which \(\omega : T \rightarrow \Omega\) is a constant mapping, then \(a\) is envy-free.

Theorem
If an economy \(e\) satisfies either Condition (A) or Condition (B), then there exists a lottery allocation for \(e\) that is both envy-free and Pareto efficient.

Proof.
- Reallocate the artificial currency equally across agents
- Reallocate all other goods to the outside endowment
- Compute a lottery equilibrium
Combinatorial allocation

A-LEEI

**Setting:** a set of goods (e.g. courses) to allocate among a set of agents (e.g. students) who demand bundles (e.g. schedules)

**A-LEEI mechanism:**

1. Ask agents to report their utility functions
2. Consider a continuum replication of the setting
3. Compute a lottery equilibrium from equal incomes, which determines a lottery for each original agent
4. Resolve lotteries and assign agents their bundles
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▶ “Approximate” because there will be some market clearing error
## Previous approaches to combinatorial allocation

### Other versions of (A)-LEEI

<table>
<thead>
<tr>
<th>Paper</th>
<th>Constraints</th>
<th>Utilities</th>
<th>Clearing error</th>
</tr>
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<tr>
<td>Hylland and Zeckhauser (1979)</td>
<td>capacity</td>
<td>unit demand</td>
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<tr>
<td>Budish, Che, Kojima, and Milgrom (2013)</td>
<td>bihierarchy</td>
<td>additive</td>
<td>none</td>
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<td>Akbarpour and Nikzad (2017)</td>
<td>general</td>
<td>additive</td>
<td>small</td>
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Previous approaches to combinatorial allocation

A-CEEI

- **A-CEEI**: approximate *competitive* equilibrium from equal incomes
  - Budish (2011)
  - Budish and Kessler (2016); Budish, Cachon, Kessler and Othman (2017)

- “Approximate” because
  - there will be some market clearing error
  - incomes cannot be perfectly equal
Social Lotteries

- In economies with non-convexities, lotteries concavify indirect utility functions
  - efficiency gains (Friedman and Savage, 1948)
  - strengthens the benefits of social insurance

- Suggests that governments should offer menus of actuarially fair “social lotteries”
  - binary lotteries would suffice
  - certain safeguards might be appropriate
Summary

- With indivisibilities and/or non-convex preferences, it can be costly to prohibit trades of probability shares of bundles
  - existence
  - first welfare theorem
  - second welfare theorem
Next steps

Investigate properties of A-LEEI:
- Bound the rate at which clearing error diminishes in finite economies as the market grows
- Empirical comparison to A-CEEI (Budish and Kessler, 2016)

Explore other applications:
- Dynamic allocation
- Two-sided matching
Back-Up Slides
Apply the A-LEEI mechanism to the $K$-fold replication of a fixed finite economy

Clearing error (as a fraction of the total supply) should be

- $O\left(\frac{1}{\sqrt{K}}\right)$ for each good, except with probability that is $O(e^{-K})$
- $O\left(\frac{1}{\sqrt{K}}\right)$ for all goods uniformly, except with probability that is $O(e^{-\frac{K}{m+n}})$
References I


