Infinite-dimensional linear programming with applications to economics

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I: Basics

I. Quick review of linear programming (LP)

$$\min_{x} \quad c^\top x$$

$$\text{subject to} \quad Ax \geq b$$

where $x \in \mathbb{R}^n$

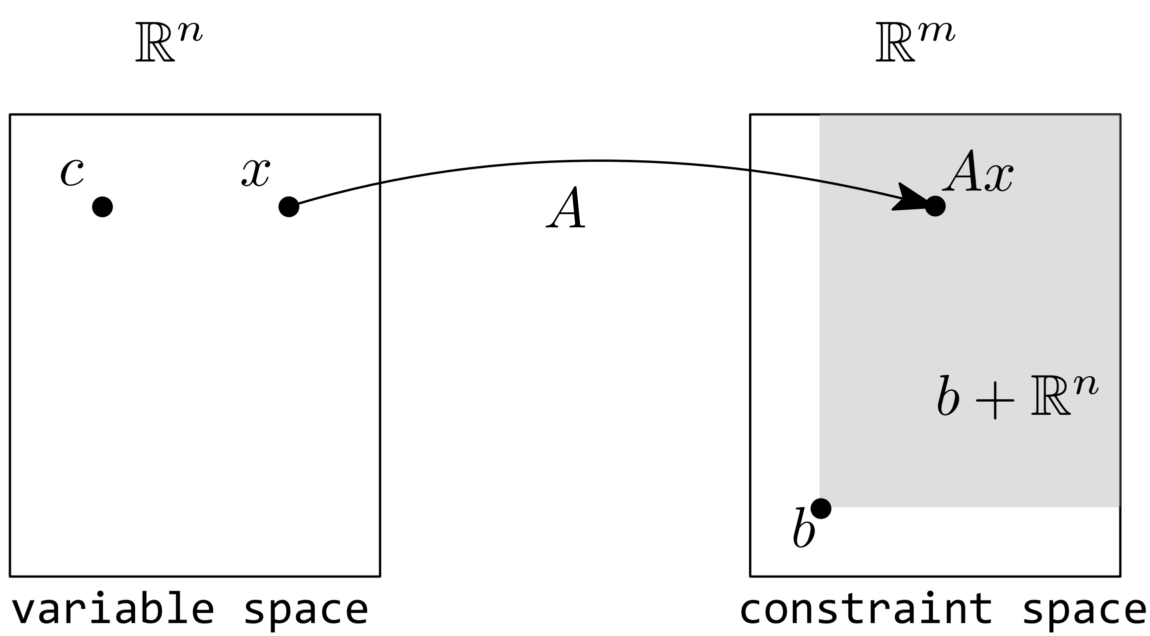
$$b \in \mathbb{R}^m$$

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

The mapping view

$$\min_x \quad c^\top x$$
$$\text{subject to} \quad Ax \geq b$$

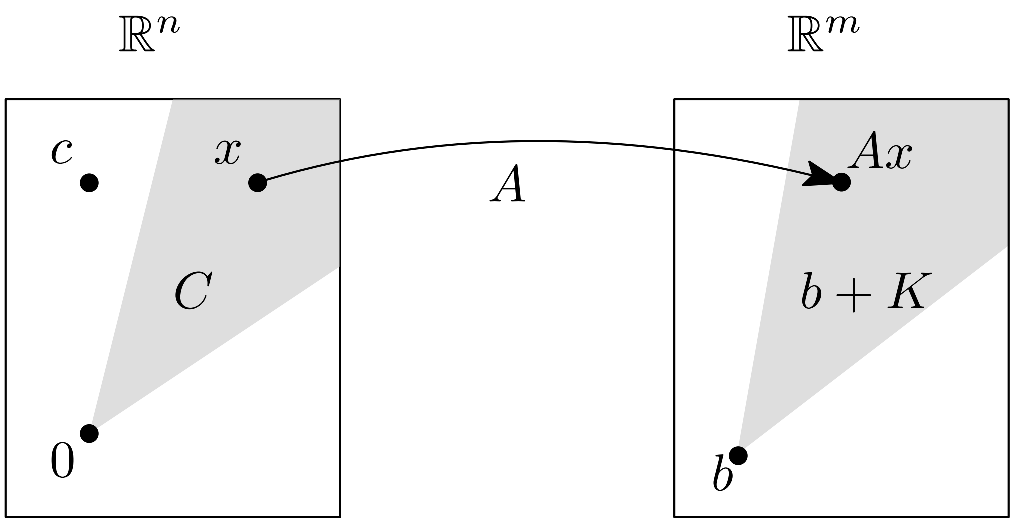
$$Ax \geq b \iff Ax \in b + \mathbb{R}^n_+$$



More generally…

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{subject to} \quad & Ax \succeq_K b \\ & x \succeq_C 0 \end{aligned}$$

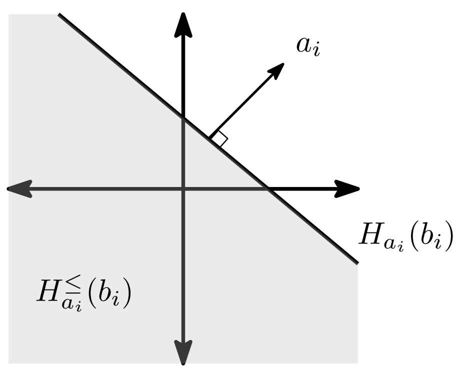
$$u \succeq_K v \iff u \in v + K$$

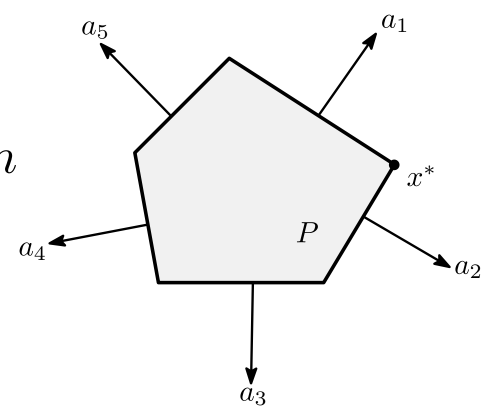


The "polyhedral" view

$$\max_x \quad c^\top x$$
$$\text{s.t.} \quad Ax \le b$$

$$Ax \le b \iff a_i^\top x \le b_i \ \ \forall i = 1, \dots, m$$





Things to like about FDLP

- If the problem is bounded, an optimal solution always exists. ✗
- When an optimal solution exists, at least one solution is an extreme point. ✗
- There exists polynomial time algorithms to find optimal solutions. ✗
- There is a nice duality theory. ?

Making things infinite

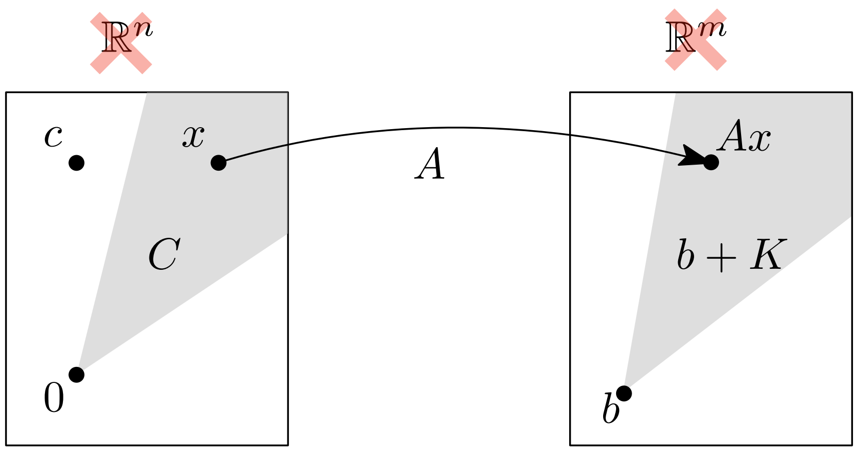
$$\begin{aligned} \min_x \quad & c^\top x \\ \text{subject to} \quad & Ax \succeq_K b \\ & x \succeq_C 0 \end{aligned}$$

Alternatives:

sequences: ℓ^p

functions: $\mathcal{L}^p(\Omega, \lambda)$ $\mathcal{C}(\Omega)$

measures: $\mathcal{M}(\Omega)$



Making sacrifices

Properties of $\mathbb{R}^n$:

- vector space structure ✓
  - > convexity
  - > extreme points
- "ball" topology ✗
- inner product, projection ✗
- separating hyperplane theory ✓

We can generalize to locally convex topological vector spaces (lctvs)

Paired vector spaces

Vector spaces X and W are paired if there exists a pairing

$$\langle \cdot, \cdot \rangle : X \times W \to \mathbb{R}$$

$$(x, w) \mapsto \langle x, w \rangle$$

such that

(P0) $\langle x, w \rangle \in \mathbb{R} \ \forall x \in X, w \in W$

(P1) $\langle \cdot, \cdot \rangle$ is bilinear

(P2) $\langle x, \bar{w} \rangle = 0 \ \forall x \in X \implies \bar{w} = 0_W$

$\langle \bar{x}, w \rangle = 0 \ \forall w \in W \implies \bar{x} = 0_X$

Some examples

(P0) $\langle x, w\rangle \in \mathbb{R}\ \forall x \in X, w \in W$

(P1) $\langle \cdot, \cdot \rangle$ is bilinear

(P2) $\langle x, \bar{w}\rangle = 0\ \forall x \in X \implies \bar{w} = 0_W$

$\langle \bar{x}, w\rangle = 0\ \forall w \in W \implies \bar{x} = 0_X$

(i) $X = W = \mathbb{R}^n$

$$\langle x, w\rangle := x^\top w = \sum_{j=1}^{n} x_j w_j$$

(ii) $X = \mathbb{R}^n, W = \mathbb{R}^m,$ with $n < m$

$$\langle x, w\rangle := x^\top w = \sum_{j=1}^{n} x_j w_j$$

$$\bar{w} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

(iii) $X = \ell^p, W = \ell^q$ $\quad \frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p, q \leq \infty$

$$\langle x, w\rangle := \sum_{j=1}^{\infty} x_j w_j$$

(Hölder’s inequality)

Some examples

(P0) $\langle x, w\rangle \in \mathbb{R}\ \forall x \in X, w \in W$

(P1) $\langle \cdot, \cdot \rangle$ is bilinear

(P2) $\langle x, \bar{w}\rangle = 0\ \forall x \in X \implies \bar{w} = 0_W$

$\langle \bar{x}, w\rangle = 0\ \forall w \in W \implies \bar{x} = 0_X$

(iv)

$$X = \mathcal{L}^p(\Omega, \lambda), W = \mathcal{L}^q(\Omega, \lambda)$$

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 \leq p, q \leq \infty$$

$$\langle x, w\rangle := \int_\Omega x(t)w(t)d\lambda(t)$$

(Hölder's inequality)

(v) $X = \mathcal{M}(\Omega), W = \mathcal{C}(\Omega)$

$$\langle x, w\rangle := \int_\Omega w dx$$

(Reisz representation theory)

Geometric interpretation

(iii) $X = \ell^p, W = \ell^q$

$$\langle x, w \rangle := \sum_{j=1}^{\infty} x_j w_j$$

(iv)

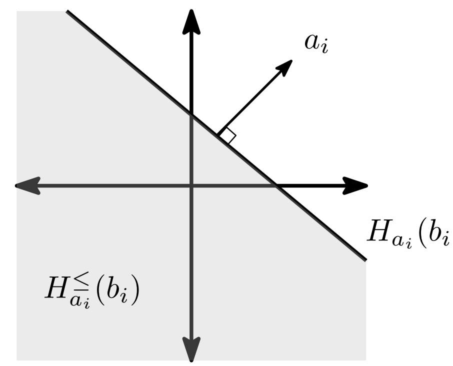
$$X = \mathcal{L}^p(\Omega, \lambda), W = \mathcal{L}^q(\Omega, \lambda)$$

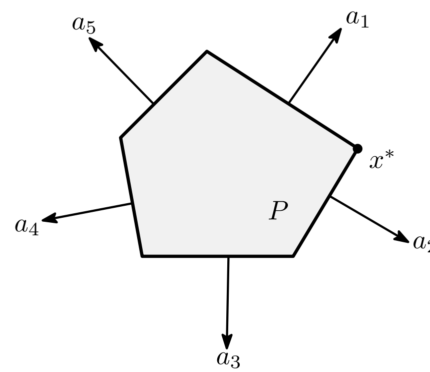
$$\langle x, w \rangle := \int_\Omega x(t) w(t) d\lambda(t)$$

(v) $X = \mathcal{M}(\Omega), W = \mathcal{C}(\Omega)$

$$\langle x, w \rangle := \int_\Omega w dx$$

$$Ax \geq b \iff a_i^\top x \geq b_i \quad \forall i = 1, \ldots, n$$





From pairing to topology, I

By (P1), the pairing $\langle \cdot, \cdot \rangle : X \times W \to \mathbb{R}$

generates linear functionals over X:

<u>Definition (algebraic dual):</u>

$$X' = \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is linear}\}$$

(P1) $\Longrightarrow J : W \to X'$

$$w \mapsto \varphi_w$$

where $\varphi_w : X \to \mathbb{R}$

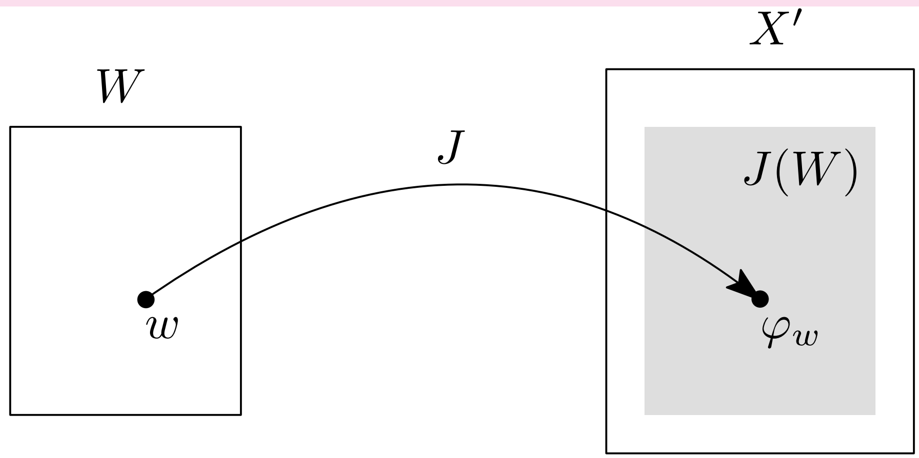
$$x \mapsto \langle x, w \rangle$$

(P2) $\Longrightarrow J$ is 1:1

$\Longrightarrow J(W) \cong$ subspace of X'

Ex. $X = W = \mathbb{R}^n$

From pairing to topology, II



"weak topology":

$$\sigma(X, W) := \{\text{smallest topology s.t. } \varphi \text{ cts } \forall \varphi \in J(W)\}$$

<u>Recall:</u> $\underset{(X,\tau)}{\varphi : X} \to \underset{(Y,\sigma)}{Y}$ cts iff $\varphi^{-1}(\mathcal{O}) \in \tau \quad \forall \mathcal{O} \in \sigma$

Topological duals

<u>Definition (topological dual):</u>

Let X be a vector space with topology τ

$$X^*_\tau := \{\varphi : X \to \mathbb{R} \mid \varphi \text{ linear and } \tau - \text{cts}\}$$

<u>Theorem:</u> Let X and W be paired vector spaces

$$X^*_{\sigma(X,W)} \cong W$$

Ex: $$(\ell^\infty)^*_{\tau_{||\cdot||_\infty}} \cong \ell_1 \oplus pfa$$

$$(\ell^\infty)^*_{\sigma(\ell^\infty,\ell^1)} \cong \ell_1$$

“Paired” linear program

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{subject to} \quad & Ax \succeq_K b \\ & x \succeq_C 0 \end{aligned}$$

Let (X,W) having pairing $\langle \cdot, \cdot \rangle$:

(PLP) “conic LP”

$$\begin{aligned} \min_{x \in X} \quad & \langle x, c \rangle \\ \text{subject to} \quad & Ax \succeq_K b \\ & x \succeq_C 0 \end{aligned}$$

$c \in W$

$A : X \to Z$

$b \in Z$

K cone in Z

C cone in X

Ex. $$Ax = \begin{bmatrix} \langle x, a_1 \rangle \\ \langle x, a_2 \rangle \\ \vdots \\ \langle x, a_m \rangle \end{bmatrix}$$ $a_i \in W$ $Z = \mathbb{R}^m$

“Paired” linear program: Mapping view

$$\min_{x \in X} \quad \langle x, c \rangle$$
$$\text{s.t.} \quad Ax \succeq_K b$$
$$x \succeq_C 0$$

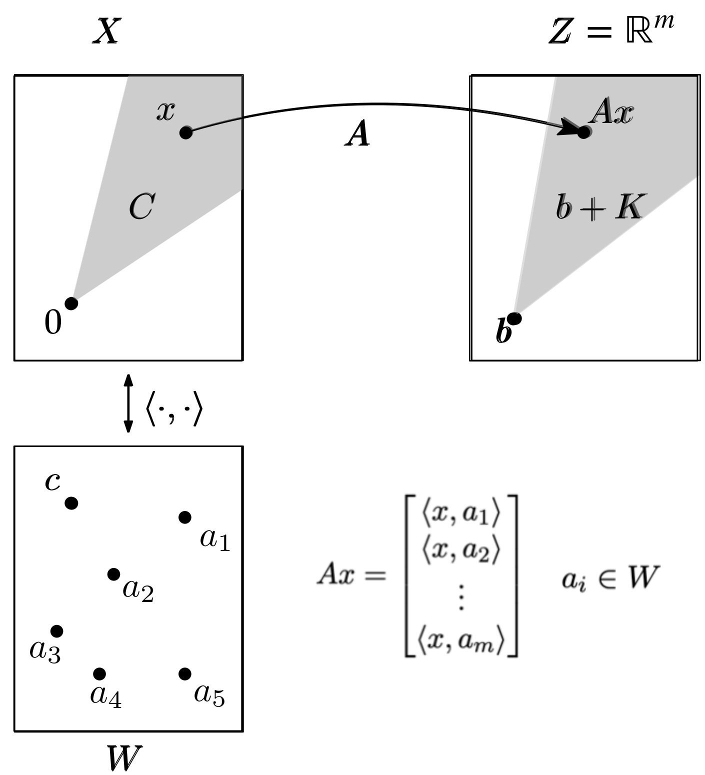
$c \in W$

$A : X \to Z$

$b \in Z$

K cone in Z

C cone in X



$$Ax = \begin{bmatrix} \langle x, a_1 \rangle \\ \langle x, a_2 \rangle \\ \vdots \\ \langle x, a_m \rangle \end{bmatrix} \quad a_i \in W$$

Countably infinite LP

$$\min_x \quad \sum_{j=1}^{\infty} x_j c_j$$

$$\text{subject to} \quad \sum_{j=1}^{\infty} x_j a_{ij} = b_i \text{ for } i = 1, \dots, m$$

$$x_j \geq 0 \text{ for } j = 1, 2, \dots$$

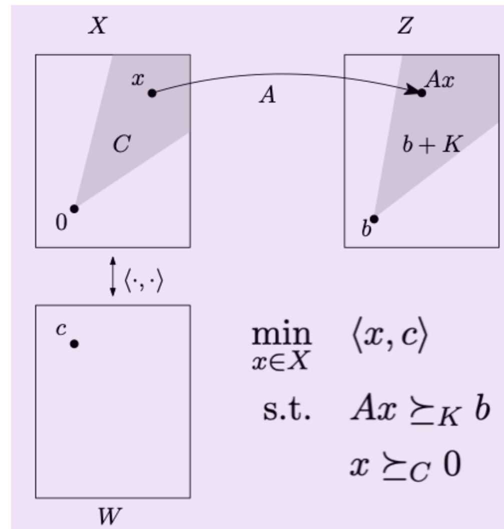
$$X = \ell^p$$

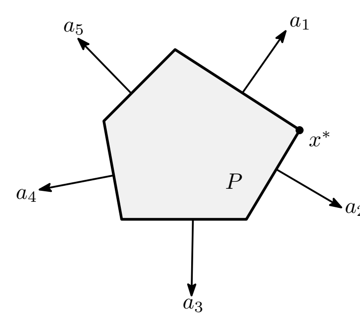
$$W = \ell^q$$

$$Z = \mathbb{R}^m$$

$$K = \{0\}$$

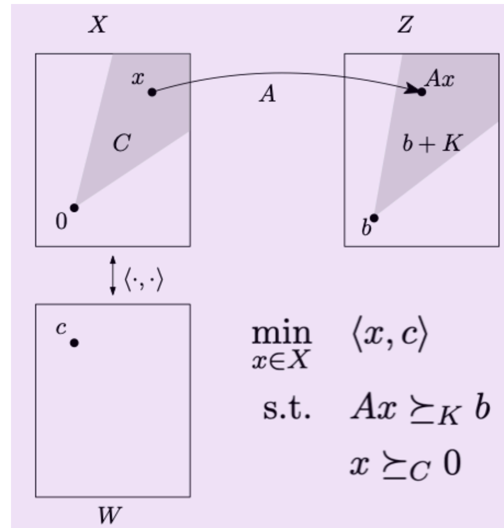
$$C = (\ell^p)_+$$





Moment problem

$$\begin{aligned} \min_{\mu} \quad & \int_{\Omega} c d\mu \\ \text{subject to} \quad & \int_{\Omega} a_i d\mu = m_i \text{ for } i = 1, 2, \ldots, q \\ & \mu \geq 0 \end{aligned}$$



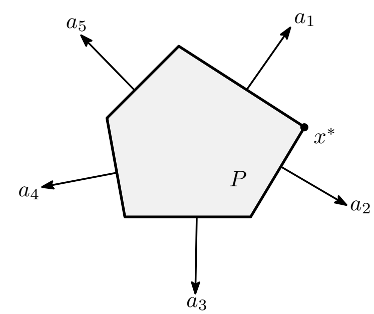
$$X = \mathcal{M}(\Omega)$$

$$W = \mathcal{C}(\Omega)$$

$$Z = \mathbb{R}^q$$

$$K = \{0\}$$

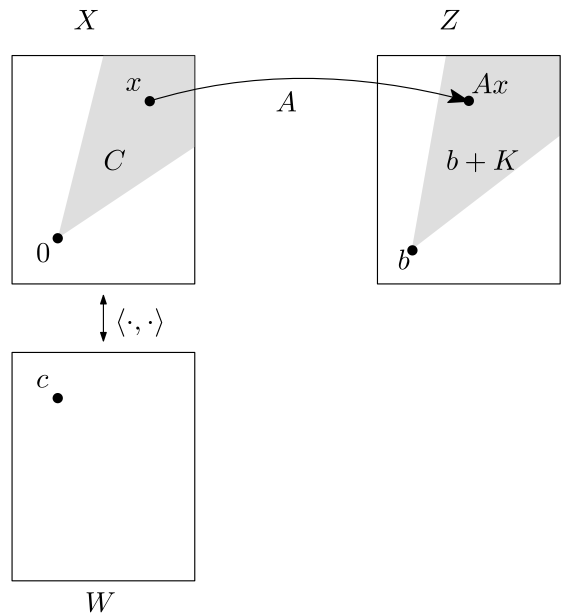
$$C = \mathscr{M}(\Omega)_+$$

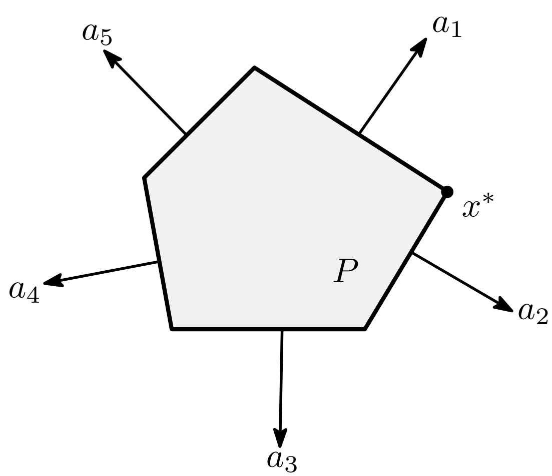


II: Existence and Extrema

Our problem

$$\min_{x \in X} \quad \langle x, c \rangle$$
$$\text{subject to} \quad Ax \succeq_K b$$
$$x \succeq_C 0$$





When does an optimal solution exist?

When an extreme point?

Weierstrass Theorem

$$\text{(P)} \quad \begin{aligned} \inf_x \quad & f(x) \\ \text{subject to} \quad & x \in F \end{aligned} \qquad f : X \to \mathbb{R} \quad (X, \tau)$$

<u>Theorem:</u> (W1) f is τ-cts
(W2) F is τ-compact } $\Longrightarrow$ (P) has an optimal solution

<u>Note:</u>
- inherent tradeoff between continuity and compactness
- weak topologies are often leveraged here
- working from compactness definition is often hopeless

Compactness theorems

Category:Compactness theorems

5 languages

Category Talk Read Edit View history Tools

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In mathematics, specifically in topology and functional analysis, compactness theorems provide necessary or sufficient conditions for the compactness of a set.

Pages in category "Compactness theorems"

The following 19 pages are in this category, out of 19 total. This list may not reflect recent changes.

A

- Arzelà–Ascoli theorem

B

- Banach–Alaoglu theorem
- Blaschke selection theorem
- Bolzano–Weierstrass theorem

C

- Cantor's intersection theorem

E

- Eberlein–Šmulian theorem

F

- Fraňková–Helly selection theorem
- Fréchet–Kolmogorov theorem

G

- Gromov's compactness theorem (topology)

H

- Heine–Borel theorem
- Helly's selection theorem

K

- Kuratowski's intersection theorem

M

- Mahler's compactness theorem
- Mazur's lemma
- Michael selection theorem
- Montel's theorem
- Mumford's compactness theorem

P

- Prokhorov's theorem

S

- Sobolev inequality

won't cover this

Banach-Alaoglu (B-A) Theorem

Let $(X, ||\cdot||_X)$ and $(W, ||\cdot||_W)$ be normed vector spaces with (X,W) paired and $W^*_{\tau_{||\cdot||_W}} \cong X$.

Then $U = \{x \in X \mid ||x||_X \leq 1\}$ is $\sigma(X,W)$-compact.

In particular,

(BA1) F is $\sigma(X,W)$-closed
(BA2) F is (norm) bounded
$\Bigg\} \Longrightarrow$ F is $\sigma(X,W)$-compact

<u>Note:</u> Reminiscent of Heine-Borel Theorem.

<u>Ex:</u> $X = \ell^\infty$, $W = \ell^1$ but the reverse does not work.

Arzelà-Ascoli (A-A) Theorem (for problems in C(Ω))

Let $X = \mathcal{C}(\Omega)$ with sup-norm topology and F a subset of X. Then:

(AA1) F is $||\cdot||_\infty$-bounded

(AA2) F is equicontinuous

$+||\cdot||_\infty$-closed

$\Bigg\} \Longleftrightarrow$ F is $||\cdot||_\infty$-compact

Each f in F has a common Lipschitz-constant M, i.e.

$$|f(\omega_1) - f(\omega_2)| \leq M|\omega_1 - \omega_2|$$

$$\forall f \in F \text{ and } \omega_1, \omega_2 \in \Omega$$

$\Longrightarrow$ F is equicontinuous

Helly's selection theorem (for problems in $L^p(\Omega)$)

Let $X = \mathcal{L}^p(\Omega)$ with its norm topology and F a subset of X. Then:

(H1) f in F are nondecreasing

(H2) f in F are uniformly bounded

$+||\cdot||_p$-closed

$\Longrightarrow$ F is $||\cdot||_p$-compact

uniformly bounded: $\exists B$ s.t. $||f||_p \leq B$ for all $f \in F$

Ex: $\Omega=[0,1]$, $p=\infty$, F is all CDFs for RV's on Ω

(H1) CDFs are nondecreasing

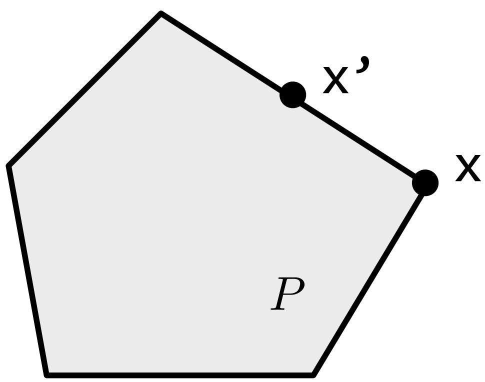
(H2) CDFs take values between 0 and 1

Extreme points & Holmes Theorem

Definition (extreme point):

$x \in F \subseteq X$ is an extreme point of F if $\nexists y, z \in F$ such that

$$x \in (y,z) := \{\alpha y + (1-\alpha)z : \alpha \in (0,1)\}$$



x is an extreme point
x' is not

Theorem (Holmes):

(X,τ) is a lctvs.*

F is τ-compact $\Longrightarrow$ F has an extreme point.

*includes both weak and norm topologies

Bauer Minimum Theorem (BMT)

$$(P) \qquad \begin{array}{ll} \inf_{x} & f(x) \\ \text{subject to} & x \in F \end{array} \qquad f : X \to \mathbb{R}$$

(X, τ)

<u>Theorem (Bauer Minimum Theorem):</u>

(BMT1) f is τ-continuous
(BMT2) f is concave
(BMT3) F is τ-compact
(BMT4) F is convex

$\Longrightarrow$ (P) has an optimal extreme point solution

"Bang-bang" control

$$\min_{x \in \mathcal{L}^\infty[0,1]} \quad \int_0^1 x(t)c(t)dt$$
$$\text{subject to} \quad \int_0^1 x(t)a_i(t)dt = b_i, \quad i = 1..m$$
$$0 \le x(t) \le 1 \text{ for a.a. } t$$
$$c, a_i \in \mathcal{L}^1[0,1]$$

$$\tau = \sigma(\mathcal{L}^\infty[0,1], \mathcal{L}^1[0,1])$$

(BMT1) f is τ-continuous ✓
(BMT2) f is concave ✓
(BMT3) F is τ-compact ✓
(BMT4) F is convex ✓
$\Longrightarrow$ (P) has an optimal extreme point solution

B-A

- Banach-Alaoglu $U = \{x \in X \mid ||x||_X \le 1\}$ is $\sigma(X, W)$-compact

$$\Longrightarrow \{x \in \mathcal{L}^\infty[0,1] \mid 0 \le x(t) \le 1\} \text{ is } \sigma(\mathcal{L}^\infty[0,1], \mathcal{L}^1[0,1]) - \text{compact}$$

- BMT $\Longrightarrow$ an optimal EP solution exists

- Fact: all extreme points have

$x(t) \in \{0,1\}$ a.a. t "bang bang"

III: Optimality of posted price

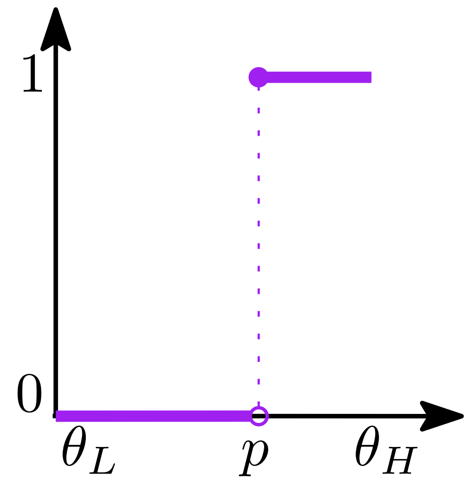
Set up (Section 2.2 of Börgers)

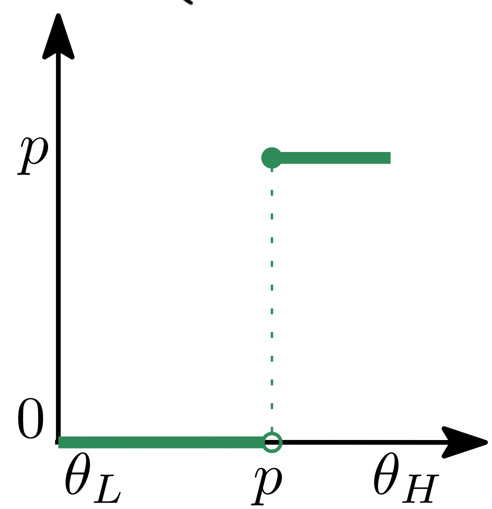
- single seller, single buyer, single indivisible good
- unknown buyer valuation θ in $[\theta_L, \theta_H]$
- θ ~ F cdf, with bounded, integrable pdf f where f(θ)>0 for all θ in $[\theta_L, \theta_H]$
- buyer has quasilinear utility: θ-t
- buyer's outside alternative normalized to 0
- seller selects a (direct) <u>mechanism</u>:
  - allocation rule, $q : [\theta_L, \theta_H] \to [0, 1]$
  - payment rule, $t : [\theta_L, \theta_H] \to \mathbb{R}$
- seller maximizes her expected payment

Posted price mechanism

$$q(\theta) := \begin{cases} 1, & \text{if } \theta \geq p \\ 0, & \text{if } \theta < p \end{cases}$$

$$t(\theta) := \begin{cases} p & \text{if } \theta \geq p \\ 0 & \text{if } \theta < p \end{cases}$$





Claim: There exists an optimal posted price mechanism.

Problem formulation

$$\max_{q,t} \quad \int_{\theta_L}^{\theta_H} t(\theta) f(\theta) d\theta$$

$$\text{s.t.} \quad \theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta') \text{ a.a. } \theta, \theta' \quad \text{(IC)}$$

$$\theta q(\theta) - t(\theta) \geq 0 \text{ a.a. } \theta \quad \text{(IR)}$$

$$0 \leq q(\theta) \leq 1 \text{ for a.a. } \theta$$

Some work:
- q increasing in $[\theta_L, \theta_H]$
- $t(\theta) = \theta q(\theta) - \int_{\theta_L}^{\theta} q(\theta) d\theta$

$$\max_{q} \quad \int_{\theta_L}^{\theta_H} \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) f(\theta) q(\theta) d\theta$$

$$\text{s.t.} \quad q \text{ increasing on } [\theta_L, \theta_H]$$

$$0 \leq q(\theta) \leq 1 \text{ for a.a. } \theta$$

How to apply our results?

$$\max_{q} \int_{\theta_L}^{\theta_H} \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) f(\theta) q(\theta) d\theta$$

$$\text{s.t.} \quad q \text{ increasing on } [\theta_L, \theta_H]$$

$$0 \le q(\theta) \le 1 \text{ for a.a. } \theta$$

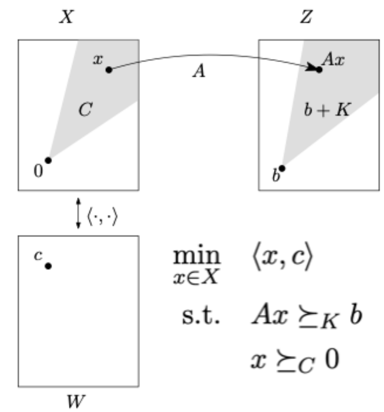
(BA1) F is $\sigma(X,W)$-closed ?
(BA2) F is (norm) bounded ✓
$\Big\} \Longrightarrow$ F is $\sigma(X,W)$-compact

Let $X = \mathcal{C}(\Omega)$ ✗ with sup-norm topology and F a subset of X. Then:

(AA1) F is $||\cdot||_\infty$-bounded
(AA2) F is equicontinuous
$\Big\} \Longleftrightarrow$ F is $||\cdot||_\infty$-compact

Let $X = \mathcal{L}^p(\Omega)$ with its norm topology and F a subset of X. Then:

(H1) f in F are nondecreasing ✓
(H2) f in F are uniformly bounded ✓
$\Big\} \Longrightarrow$ F is $||\cdot||_p$ -compact



Theorem (Bauer Minimum Theorem):

(BMT1) f is τ-continuous ✓
(BMT2) f is concave ✓
(BMT3) F is τ-compact ?
(BMT4) F is convex ✓
$\Big\} \Longrightarrow$ (P) has an optimal extreme point solution

Last insight: extreme points are “bang-bang”

Optimality of posted prices

Powerful modelling paradigm

$$\max_{q,t} \int_{\theta_L}^{\theta_H} t(\theta) f(\theta) d\theta$$

$$\text{s.t.} \quad \theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta') \text{ a.a. } \theta, \theta' \quad \text{(IC)}$$

$$\theta q(\theta) - t(\theta) \geq 0 \text{ a.a. } \theta \quad \text{(IR)}$$

$$0 \leq q(\theta) \leq 1 \text{ for a.a. } \theta$$

Tricks and insights

$$\max_{q} \int_{\theta_L}^{\theta_H} \theta q(\theta) f(\theta) d\theta - \int_{\theta_L}^{\theta_H} q(\theta) d\theta$$

$$\text{s.t.} \quad q \text{ increasing on } [\theta_L, \theta_H]$$

$$0 \leq q(\theta) \leq 1 \text{ for a.a. } \theta$$

Knowledge of what is possible

Theorem (Bauer Minimum Theorem):

(BMT1) f is τ-continuous ✓
(BMT2) f is concave ✓
(BMT3) F is τ-compact ✓
(BMT4) F is convex ✓

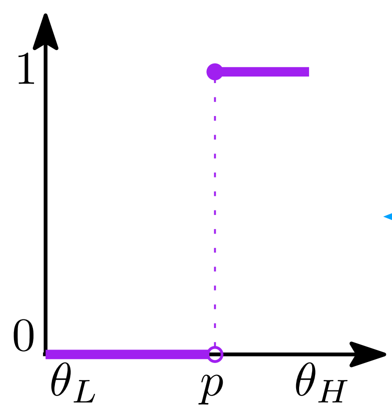
$\Longrightarrow$ (P) has an optimal extreme point solution

Let $X = \mathcal{L}^p(\Omega)$ with its norm topology and F a subset of X. Then:

(H1) f in F are nondecreasing ✓
(H2) f in F are uniformly bounded ✓

$\Longrightarrow$ F is $||\cdot||_p$ -compact

Knowing your target



"Why"?

IV: Duality theory

Refresher on FDLP duality

(P) $\min_{x \in \mathbb{R}^n} \quad c^\top x$

$\text{s.t.} \quad Ax \geq b$

$x \geq 0$

(D) $\max_{y \in \mathbb{R}^m} \quad b^\top y$

$\text{s.t.} \quad A^\top y \leq c$

$y \geq 0$

Idea: dual "linearly combines" constraints to find the "best" lower bound on (P)'s objective value implied by the constraints:

$$c^\top x \geq (A^\top y)^\top x \geq b^\top y$$

> y is in $\mathbb{R}^m$ since there are m constraints

> Ax combines columns, $A^\top y$ combines rows

> $y \geq 0$ to keep the inequalities in the right direction

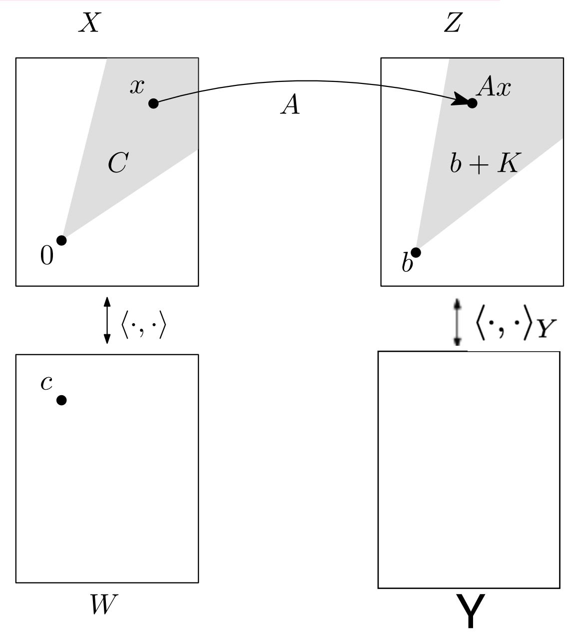
> $c \geq A^\top y$ so that we guarantee lower bounds

> $b^\top y$ is the implication of the aggregated constraint, we want to maximize

Constructing the dual of (PLP)

$$\min_{x \in X} \quad \langle x, c \rangle$$
$$\text{s.t.} \quad Ax \succeq_K b$$
$$x \succeq_C 0$$

$$\max_{y \in \mathbb{R}^m} \quad b^\top y$$
$$\text{s.t.} \quad A^\top y \leq c$$
$$y \geq 0$$



<u>Idea</u>: dual "linearly combines" constraints to find the "best" lower bound on (P)'s objective value implied by the constraints

> linearly act on the constraint space Z
> how to keep the constraints going in the right direction?
> how to guarantee valid lower bounds?

(Y,Z) paired according to pairing $\langle \cdot, \cdot \rangle_Y$

(Topological) dual cones

Definition (dual cone):

(Y,Z) paired vector spaces. K is a cone in Z. The dual cone is:

$$K^* := \{y \in Y \mid \langle y, z\rangle \geq 0 \text{ for all } z \in K\}$$

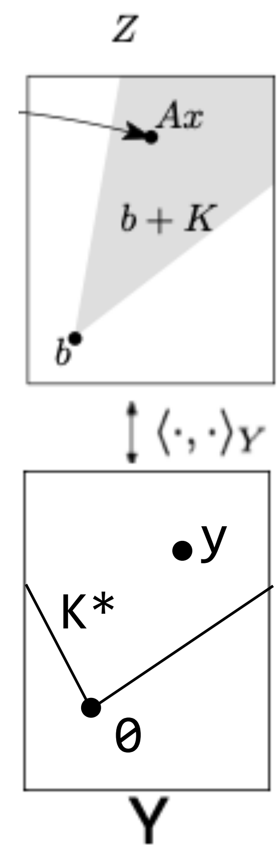
To keep constraints

$$Ax \succeq_K b$$

in right direction we need:

$$y \in K^* \iff y \succeq_{K^*} 0_Y$$

$$\max_{y \in \mathbb{R}^m} \; b^\top y \quad \text{s.t.} \quad A^\top y \leq c, \quad y \geq 0$$



(Topological) adjoint

<u>Definition (adjoint):</u>

Let (X,W) and (Y,Z) be paired spaces with pairings $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$.

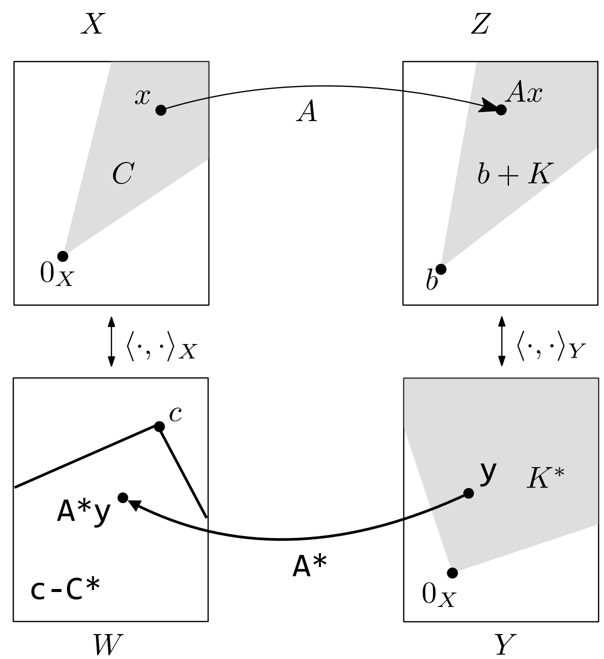
Let $A : X \to Y$ be $\sigma(X, W) - \sigma(Z, Y)$ continuous.

Then the adjoint

$$A^* : Y \to W$$

exists where

$$\langle x, A^* y \rangle_X = \langle y, Ax \rangle_Y$$



$$\max_{y \in \mathbb{R}^m} \quad b^\top y$$
$$\text{s.t.} \quad A^\top y \leq c$$
$$y \geq 0$$

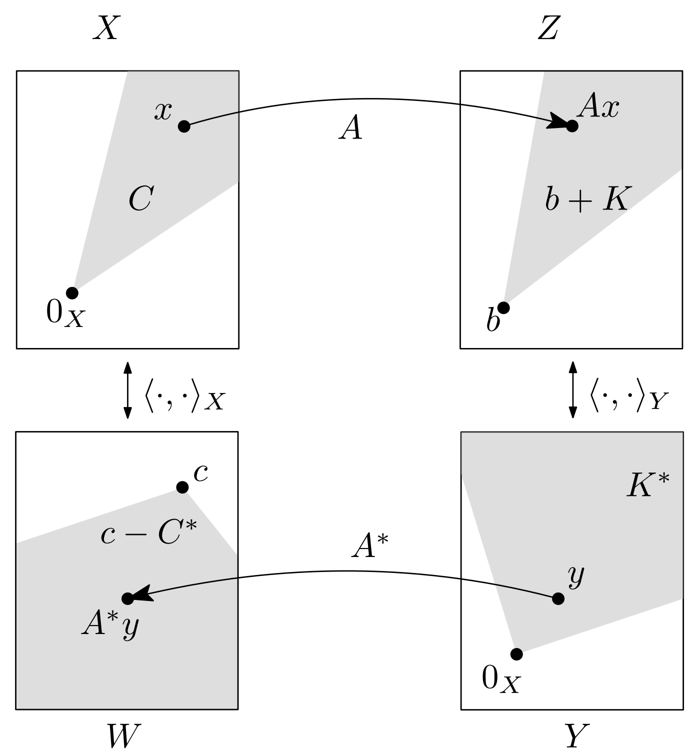
$$A^* y \leq_C^* c \iff A^* y \in c - C^*$$

(PLP) duality

Let (X,W) and (Y,Z) be paired spaces with pairings $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$.
Let $A: X \to Y$ be $\sigma(X,W) - \sigma(Z,Y)$ continuous.

(PLP)
$$\begin{aligned} \min_{x\in X} \quad & \langle x, c\rangle_X \\ \text{s.t.} \quad & Ax \succeq_K b \\ & x \succeq_C 0 \end{aligned}$$

(PLPD)
$$\begin{aligned} \max_{y\in Y} \quad & \langle y, b\rangle_Y \\ \text{s.t.} \quad & A^* y \preceq_{C^*} c \\ & y \succeq_{K^*} 0 \end{aligned}$$



CILP duality

$$\min_x \quad \sum_{j=1}^{\infty} x_j c_j$$
$$\text{s.t.} \quad \sum_{j=1}^{\infty} x_j a_{ij} = b_i \text{ for } i = 1, 2, \ldots$$
$$x_j \geq 0 \text{ for } j = 1, 2, \ldots$$

$X = \ell^p$

$W = \ell^q$

$Z = \ell^\infty$

$Y = \ell^1$

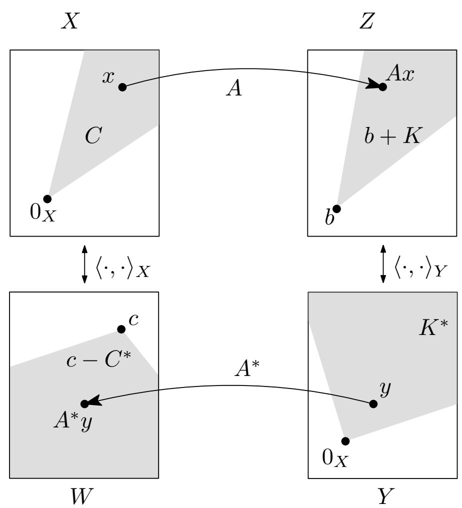
$K = \{0\}$

$K^* = (\ell^1)_+$

$C = (\ell^p)_+$

$C^* = (\ell^q)_+$

$$A^* y = \sum_{i=1}^{\infty} y_i a_{ij}$$



$$\max_y \sum_{i=1}^{\infty} y_i b_i$$
$$\text{s.t.} \quad \sum_{i=1}^{\infty} y_i a_{ij} \leq c_j \text{ for } j = 1, 2, \ldots$$
$$y_i \geq 0 \text{ for } i = 1, 2, \ldots$$

$\langle x, A^* y\rangle_X = \langle y, Ax\rangle_Y$

$K^* := \{y \in Y \mid \langle y, z\rangle \geq 0 \text{ for all } z \in K\}$

Assuming: $\sup_i \left\{ \sum_{j=1}^{\infty} |a_{ij}| \right\} < \infty$

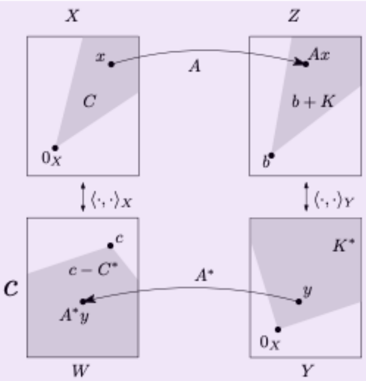
Duality results

(PLP)

$$\min_{x\in X} \quad \langle x, c\rangle_X \qquad \text{s.t.} \quad Ax \succeq_K b, \quad x \succeq_C 0$$

(PLPD)

$$\max_{y\in Y} \quad \langle y, b\rangle_Y \qquad \text{s.t.} \quad A^*y \preceq_{C^*} c, \quad y \succeq_{K^*} 0$$



<u>Theorem (weak duality)</u>

val(PLPD) ≤ val(PLP)

<u>Theorem (complementary slackness):</u>

$$\left.\begin{array}{r}\bar{x} \text{ optimal to } (PLP)\\ \bar{y} \text{ optimal to } (PLPD)\\ \langle \bar{x}, c\rangle_X = \langle \bar{y}, b\rangle_Y\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{r}\langle \bar{x}, c - A^*(\bar{y})\rangle_X = 0\\ \langle \bar{y}, A\bar{x} - b\rangle_Y = 0\end{array}\right.$$

Workhorse of IDLP story-telling:

CS conditions + extreme point structure

Unlocked by showing “zero-duality gap”

A summary of zero-duality gap results

- (Anderson and Nash, 1987), (Barvinok,2002)
  topology of epigraphical cones
  - closedness
  - interior point
  - boundedness
  - compactness

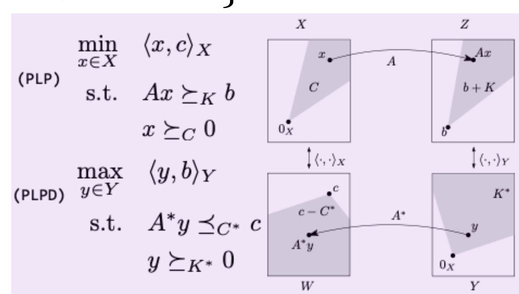
$$\hat{A}(C) := \{(Ax, \langle x, c\rangle_X) \mid x \in C\} \subseteq Z \times \mathbb{R}$$

- (Shapiro, 2001), (Rockafellar, 1974)

  topology of optimal value functional

$$v(z) := \min\{\langle x, c\rangle \mid x \in C, Ax + z \in K\}$$

  - subdifferentiability



(Generalized) Slater condition

Theorem (Slater condition): $\mathrm{val}(PLP) > -\infty$

$\exists \bar{x} \in C \text{ s.t. } A\bar{x} - b \in \mathrm{int}_{\sigma(Z,Y)} K \Longleftrightarrow$
- val(PLP) = val(PLPD)
- (PLPD) has an optimal solution

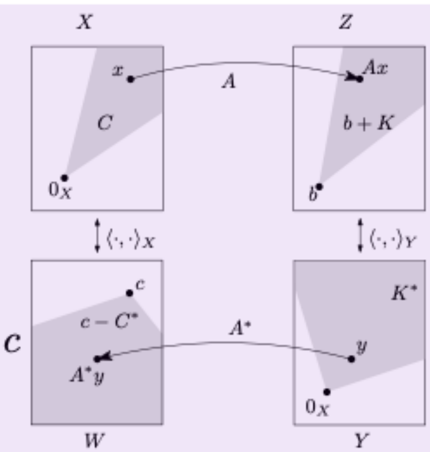
Road map:

(i) Apply BMT (using a heavy compactness result) for primal existence and extreme points

(ii) Show zero-duality gap and dual existence using Slater, or scramble for tricks

(PLP) $\min_{x \in X} \langle x, c \rangle_X$ s.t. $Ax \succeq_K b$, $x \succeq_C 0$

(PLPD) $\max_{y \in Y} \langle y, b \rangle_Y$ s.t. $A^* y \preceq_{C^*} c$, $y \succeq_{K^*} 0$



(iii) Apply CS to further analyze the extremal structure.

V: Linear persuasion

The linear persuasion model

(based on Dizdar and Kováč, GEB, 2020)

- sender influences the beliefs of a receiver through deciding how to reveal information
- state of the world S distributed according to Borel probability measure μ
- Assume supp(μ) in [0,1], including {0,1}
- Sender utility $u : [0,1] \to \mathbb{R}$ depend's only on the mean of the receiver's posterior beliefs τ
- τ is derived by Bayesian updating from prior μ, this updating depends on sender's choice
- Upshot: τ is feasible iff

$$\tau \preceq_{cx} \mu \iff \int_0^1 v d\tau \leq \int_0^1 v d\mu \quad \forall v : [0,1] \to \mathbb{R} \text{ cvx, cts}$$

* a new proof of (Dworczak and Martini, JPE, 2019)

Formulation as a (PLPD)!

$$\max_{\tau} \quad \int_0^1 u\,d\tau$$
$$\text{s.t.} \quad \tau \preceq_{cx} \mu$$
$$\tau \succeq 0$$
$$\int_0^1 d\tau = 1$$

Definition (dual cone):

(Y,Z) paired vector spaces. K is a cone in Z. The dual cone is:

$$K^* := \{y \in Y \mid \langle y, z\rangle \geq 0 \text{ for all } z \in K\}$$

$$\tau \preceq_{cx} \mu \iff \int_0^1 v\,d\tau \leq \int_0^1 v\,d\mu$$

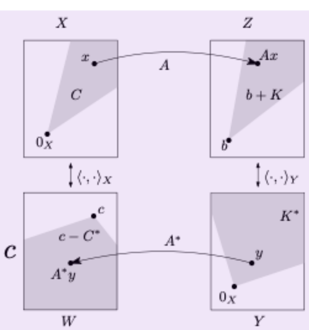
$$U[0,1] : \forall v : [0,1] \to \mathbb{R} \text{ cvx, cts}$$

$$0 \preceq_{cx} \tau \iff 0 \leq \int_0^1 v\,d\tau$$

$Y = \mathcal{M}[0,1]$	$A^*\tau = \tau$
$Z = \mathcal{C}[0,1]$	$Ax = x$
$b = u$	$K^* = \mathcal{M}[0,1]_+$
$c = \mu$	$K = \mathcal{C}[0,1]_+$
$W = \mathcal{M}[0,1]$	$C^* = U[0,1]^*$
$X = \mathcal{C}[0,1]$	$C = U[0,1]$

(PLP) $\quad \min_{x \in X} \ \langle x, c\rangle_X \quad \text{s.t.} \ Ax \succeq_K b, \ x \succeq_C 0$

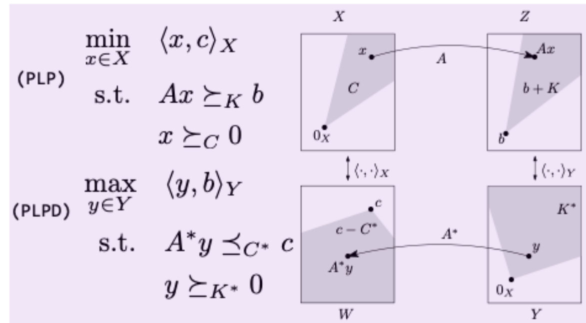
(PLPD) $\quad \max_{y \in Y} \ \langle y, b\rangle_Y \quad \text{s.t.} \ A^*y \preceq_{C^*} c, \ y \succeq_{K^*} 0$



Formulation as a (PLP)

$$
\begin{aligned}
Y &= \mathcal{M}[0,1] & A^*\tau &= \tau \\
Z &= \mathcal{C}[0,1] & Ax &= x \\
b &= u & K^* &= \mathcal{M}[0,1]_+ \\
X &= \mathcal{C}[0,1] & K &= \mathcal{C}[0,1]_+ \\
W &= \mathcal{M}[0,1] & C^* &= U[0,1]^* \\
c &= \mu & C &= U[0,1]
\end{aligned}
$$

$U[0,1] : \forall v : [0,1] \to \mathbb{R}$ cvx, cts



(P) $\min_{p\in\mathscr{C}[0,1]} \langle p, \mu\rangle$ s.t. $Ap \succeq_{\mathcal{C}[0,1]_+} u$, $p \succeq_{U[0,1]} 0$

(D) $\max_{\tau\in\mathcal{M}[0,1]} \langle u, \tau\rangle$ s.t. $A^*\tau \preceq_{U[0,1]^*} \mu$, $\tau \succeq_{\mathcal{M}[0,1]_+} 0$

(i) Apply BMT (using a heavy compactness result)

(ii) Show ZDG using Slater or tricks

(iii) Apply CS for structure

Applying BMT to (P)

(P) $\min_{p \in \mathscr{C}[0,1]} \langle p, \mu \rangle$

s.t. $p \succeq_{\mathcal{C}[0,1]_+} u$

$p \succeq_{U[0,1]} 0$

<u>Idea:</u>

- Without loss of optimality, p should not be much bigger than u.
- p is convex and continuous so p' exists a.e. and must be uniformly bounded
- p is thus uniformly bounded and Lipschitz with same constant, which implies equicts
- norm compact implies weak compact

$\tau = \sigma(\mathcal{C}[0,1], \mathcal{M}[0,1])$

(BMT1) f is τ-continuous ✓
(BMT2) f is concave ✓
(BMT3) F is τ-compact ? ✓
(BMT4) F is convex ✓
$\Longrightarrow$ (P) has an optimal extreme point solution

(BA1) F is $\sigma(X,W)$-closed ?
(BA2) F is (norm) bounded
$\Longrightarrow$ F is $\sigma(X,W)$-compact

Let $X = \mathcal{C}(\Omega)$ with sup-norm topology and F a subset of X. Then:

(AA1) F is $\|\cdot\|_\infty$-bounded ✓
(AA2) F is equicontinuous ✓
$\Longleftrightarrow$ F is $\|\cdot\|_\infty$-compact ✓

Let $X = \mathcal{L}^p(\Omega)$ with its norm topology and F a subset of X. Then:

(H1) f in F are nondecreasing ✗
(H2) f in F are uniformly bounded
$\Longrightarrow$ F is $\|\cdot\|_p$ -compact

* This is the approach of Dizdar and Kováč.

Applying BMT to (D)

$\tau = \sigma(\mathcal{M}[0,1], \mathcal{C}[0,1])$

* This is the approach of Kleiner, Moldovanu, and Strack, ECMA, 2021.

$$\text{(D)} \quad \max_{\tau \in \mathcal{M}[0,1]} \langle u, \tau \rangle \quad \text{s.t.} \quad \tau \preceq_{U[0,1]^*} \mu, \quad \tau \succeq_{\mathcal{M}[0,1]_+} 0$$

Idea:

- Reformulate the problem in terms of the CDF functions of τ.
- μ ~ F CDF, and τ ~ G CDF
- F,G:[0,1] → [0,1], nondecreasing, right-cts, in L1
- $G \succ F$ if $\int_t^1 G(s)ds \geq \int_t^1 F(s)ds$, $\int_0^1 G(s)ds = \int_0^1 F(s)ds$

"mean-preserving spread/contraction"

(BMT1) f is τ-continuous ✓
(BMT2) f is concave ✓
(BMT3) F is τ-compact ?
(BMT4) F is convex ✓
$\Longrightarrow$ (P) has an optimal extreme point solution

(BA1) F is $\sigma(X,W)$-closed ?
(BA2) F is (norm) bounded
$\Longrightarrow$ F is $\sigma(X,W)$-compact

Let $X = \mathcal{C}(\Omega)$ ✗ with sup-norm topology and F a subset of X. Then:

(AA1) F is $\|\cdot\|_\infty$-bounded
(AA2) F is equicontinuous
$\Longleftrightarrow$ F is $\|\cdot\|_\infty$-compact

Let $X = \mathcal{L}^p(\Omega)$? ✓ with its norm topology and F a subset of X. Then:

(H1) f in F are nondecreasing ✓
(H2) f in F are uniformly bounded
$\Longrightarrow$ F is $\|\cdot\|_p$ -compact

- Key observations is:

$$\tau \preceq_{cx} \mu \iff G \succ F$$

Reformulated problem

(D) $\max_{\tau \in \mathcal{M}[0,1]} \langle u, \tau \rangle$ → $\max_{G \in \mathcal{L}^1[0,1]} \int u dG$

s.t. $\tau \preceq_{U[0,1]^*} \mu$ → $s.t.\ \ G \succ F$

$\tau \succeq_{\mathcal{M}[0,1]_+} 0$ → G nondecreasing, $0 \leq G(t) \leq 1$ a.a. t

(BMT1) f is τ-continuous ✓
(BMT2) f is concave ✓
(BMT3) F is τ-compact ? ✓
(BMT4) F is convex ✓
$\Longrightarrow$ (P) has an optimal extreme point solution ✓

Let $X = \mathcal{L}^p(\Omega)$ with its norm topology and F a subset of X. Then:

(H1) f in F are nondecreasing ✓
(H2) f in F are uniformly bounded ✓
$\Longrightarrow$ F is $||\cdot||_p$ -compact ✓

<u>Remains to show:</u>

$$\{G : G \succ F\}$$

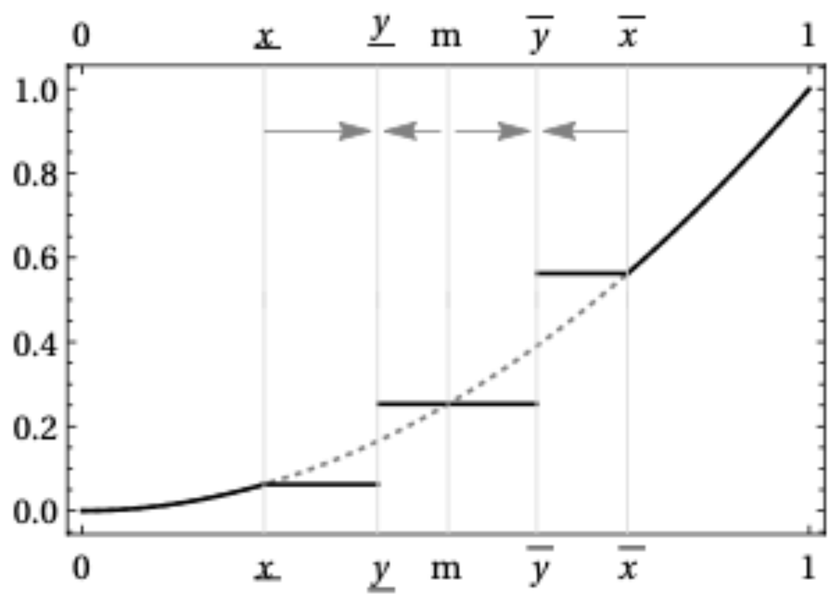
is closed in the $\mathcal{L}^1$ topology.

<u>Note:</u>

$G_n \to \hat{G}$ and $G_n \succ F \Longrightarrow \hat{G} \succ F$

$G \succ F$ if $\int_t^1 G(s)ds \geq \int_t^1 F(s)ds$, $\int_0^1 G(s)ds = \int_0^1 F(s)ds$

Structure of extreme points



G is an "ironing" of F

Implications for the structure of some optimal information design strategies

(i) Apply BMT (using a heavy compactness result)

(ii) Show ZDG using Slater or tricks

(iii) Apply CS for structure

Strong duality

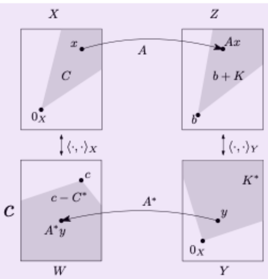
(P)
$$\max_{p \in \mathcal{C}[0,1]} \langle p, \mu \rangle \quad \text{s.t.} \quad p \succeq_{\mathcal{C}[0,1]_+} u, \quad p \succeq_{U[0,1]} 0$$

(D)
$$\max_{\tau \in \mathcal{M}[0,1]} \langle u, \tau \rangle \quad \text{s.t.} \quad \tau \preceq_{U[0,1]^*} \mu, \quad \tau \succeq_{\mathcal{M}[0,1]_+} 0$$

$$Y = \mathcal{M}[0,1] \qquad A^*\tau = \tau$$
$$Z = \mathcal{C}[0,1] \qquad Ax = x$$
$$b = u \qquad K^* = \mathcal{M}[0,1]_+$$
$$X = \mathcal{C}[0,1] \qquad K = \mathcal{C}[0,1]_+$$
$$W = \mathcal{M}[0,1] \qquad C^* = U[0,1]^*$$
$$c = \mu \qquad C = U[0,1]$$
$$U[0,1] : \forall v : [0,1] \to \mathbb{R} \text{ cvx, cts}$$

(PLP)
$$\min_{x \in X} \langle x, c \rangle_X \quad \text{s.t.} \quad Ax \succeq_K b, \quad x \succeq_C 0$$

(PLPD)
$$\max_{y \in Y} \langle y, b \rangle_Y \quad \text{s.t.} \quad A^*y \preceq_{C^*} c, \quad y \succeq_{K^*} 0$$



Theorem (Slater condition): $\text{val}(PLP) > -\infty$

$\exists \bar{x} \in C$ s.t. $A\bar{x} - b \in \text{int}_{\sigma(Z,Y)} K \iff$
- val(PLP) = val(PLPD)
- (PLPD) has an optimal solution

$$\exists ? \bar{p} \in \mathcal{C}[0,1] \cap U[0,1] \text{ s.t. } p - u \in \text{int}\mathcal{C}[0,1]_+$$

- u is continuous on [0,1] therefore bounded, by say, B
- set p(t) = B + 1 for all t, so, p - u is constant fn 1
- that function is in $\text{int}\mathcal{C}[0,1]_+$

Final step

(i) Apply BMT (using a heavy compactness result) ✓ | (ii) Show ZDG using Slater or tricks ✓ | (iii) Apply CS for structure ✓

Theorem (complementary slackness):

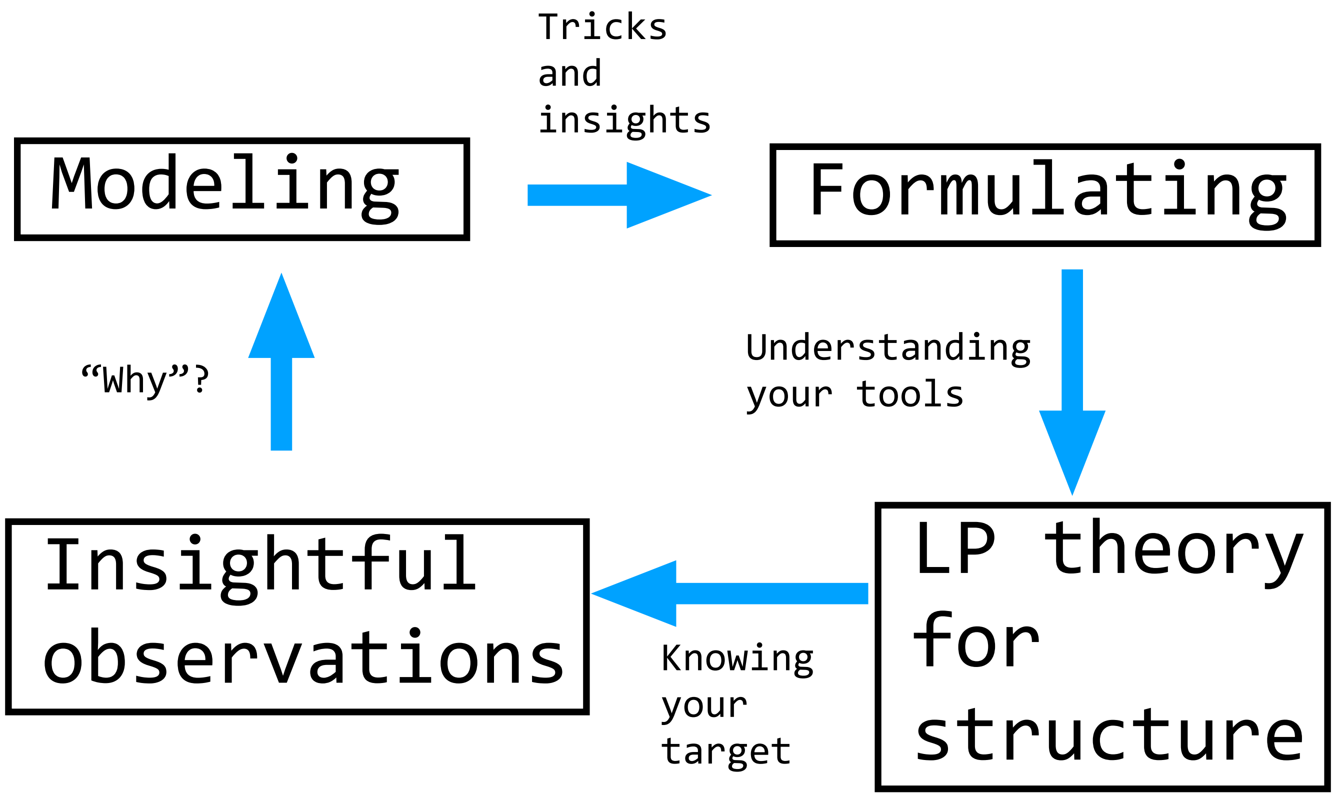
$$\left.\begin{array}{r} \checkmark\ \bar{x} \text{ optimal to } (PLP) \\ \checkmark\ \bar{y} \text{ optimal to } (PLPD) \\ \checkmark\ \langle \bar{x}, c\rangle_X = \langle \bar{y}, b\rangle_Y \end{array}\right\} \iff \left\{\begin{array}{r} \langle \bar{x}, c - A^*(\bar{y})\rangle_X = 0\ \checkmark \\ \langle \bar{y}, A\bar{x} - b\rangle_Y = 0 \end{array}\right.$$

$$\langle \hat{p}, \mu - \tau^* \rangle = 0$$

$$\langle \tau^*, \hat{p} - u \rangle = 0$$

Dworczak and Martini craft insights based on this and related facts.

<u>Note:</u> they can relax the assumption u is cts.


Modeling
Tricks and insights
Formulating
Understanding your tools
LP theory for structure
Knowing your target
Insightful observations
“Why”?

Thank you for listening !