Stable Assignments and Search Frictions

Stephan Lauermann

Universität Bonn

Georg Nöldeke

Universität Basel

April 30, 2021

1. Introduction

What we do

- We embed a static assignment problem with transferable utility into a dynamic search model ...
  - in such a way that we can identify steady-state equilibrium outcomes of the dynamic model with feasible outcomes of the static assignment problem.
- We investigate the limits of steady-state equilibrium outcomes as the velocity of the search technology goes to infinity ...
  - and ask whether such limit outcomes correspond to stable outcomes of the underlying static assignment problem.

1. Introduction

Why we do it

Investigate intuition that stable assignments are a shortcut to model situations in which frictions are negligible.

What has been done before

- Convergence to competitive equilibria in dynamic matching and bargaining games:
  Gale (JET 1987), Rubinstein and Wolinsky (Econometrica 1985), Lauermann (AER 2013), Cho and Matsui (JET 2017)
- Convergence to stable matchings in the marriage problem (NTU):
  Adachi (JET 2003), Lauermann and Nöldeke (JET 2014)

2. Assignment Problem

Assignment problem given by (B, S, v, f):

- B and S: disjoint, non-empty and finite set of agent types (buyers and sellers)
  - $T = B \cup S$
- $v(b, s)$: value of a match between a buyer of type b and a seller of type s
  - Value of staying single/unmatched is normalized to zero
  - Transferable utility
- $f(t) > 0$: mass of agents with type $t \in T$

2. Assignment Problem

- Feasible assignment: $x : B \times S \to \mathbb{R}$ satisfying

$$x(b,s) \geq 0 \text{ for all } (b,s) \in B \times S$$
$$x(t,t) \geq 0 \text{ for all } t \in T$$

where

$$x(b,b) = f(b) - \sum_{s \in S} x(b,s)$$
$$x(s,s) = f(s) - \sum_{b \in B} x(b,s)$$

- Optimal assignment solves

$$\max_{x \text{ feasible}} V(x) = \sum_{b \in B} \sum_{s \in S} x(b,s) v(b,s)$$

2. Assignment Problem

- Feasible outcome (x,u): a feasible assignment x together with a payoff profile $u : T \to \mathbb{R}$ satisfying

$$\sum_{t \in T} f(t)u(t) = V(x)$$

- A feasible outcome is individually rational if

$$u(t) \geq 0, \text{ for all } t \in T$$

  and pairwise stable if

$$u(b) + u(s) \geq v(b,s) \text{ for all } (b,s) \in B \times S$$

- A feasible outcome is stable if it is individually rational and pairwise stable

2. Assignment Problem

Recall basic results:

1. Optimal assignments and stable outcomes exist
2. If (x, u) is stable, then x is optimal

Assumption 1

- *There exists* (b, s) *such that* $v(b, s) > 0$
- $v(b, s) > 0 \Rightarrow v(b, s') \neq v(b, s)$ *and* $v(b', s) \neq v(b, s)$ *for all* $b \neq b'$ *and* $s \neq s'$

3. Search

Framework

- Random-search model in continuous time
- Mass $f(t) > 0$ of agents of each type t are “born” and enter the market per unit time
- Market is in steady-state with mass $F(t) > 0$ of agents of type t searching for a partner
- At rate $\delta > 0$ agents are exogenously removed from the market and become single with payoff of zero
- Meetings between agents are generated by a quadratic search technology with velocity parameter $\lambda > 0$:

$$\lambda F(b)F(s) > 0$$

  is the mass of agents of type b that meet agents of type s per unit time

3. Search
Framework

- When two agents meet:
  - they observe each other's type
  - each agent is selected with probability 0.5 to make a proposal for the division of $v(b,s)$; the other agent accepts or rejects
  - if the proposal is accepted, both agents leave the market and receive their agreed shares of $v(b,s)$
  - if the proposal is rejected, both agents continue to search
- Agents are risk neutral and there is no (further) discounting
- Remarks:
  - Framework as in Shimer and Smith (Econometrica 2000)
  - Quadratic search technology is an innocent simplification (Lauermann, Nöldeke, Tröger, Econometrica 2020)

3. Search

Steady state equilibrium

- Let $\alpha : B \times S \to [0,1]$ specify the (stationary) fractions $\alpha(b,s)$ of meetings between agents with types b and s that result in a match
- Payoff profile u specifies the expected payoffs of those agents who are currently searching for a partner
- A (steady-state) equilibrium is a triple (α, F, u) satisfying
  1. Inflows and outflows balance for all types
  2. For the given u, the specification of α is consistent with (subgame perfect) equilibrium in the induced bargaining games
  3. Expected payoffs solve the appropriate value equations

3. Search

Definition 1 (Equilibrium)

(α, F, u) is an equilibrium if for all b and s:

$$f(b) = F(b)[\delta + \lambda \sum_{s \in S} \alpha(b,s) F(s)] \tag{1a}$$

$$f(s) = F(s)[\delta + \lambda \sum_{b \in B} \alpha(b,s) F(b)] \tag{1b}$$

$$\alpha(b,s) = \begin{cases} 0 & \text{if } u(b) + u(s) > v(b,s) \\ 1 & \text{if } u(b) + u(s) < v(b,s) \end{cases} \tag{2}$$

$$\delta u(b) = \sum_{s \in S} \lambda F(s) \max\{0, v(b,s) - u(b) - u(s)\}/2 \tag{3a}$$

$$\delta u(s) = \sum_{b \in B} \lambda F(b) \max\{0, v(b,s) - u(b) - v(s)\}/2 \tag{3b}$$

3. Search

Equilibrium Outcomes

- Every equilibrium (α, F, u) induces an equilibrium outcome (x, u), which describes what happens to a "cohort" of agents entering the market
- u is the solution to the equilibrium value conditions and

$$x(b,s) = \lambda \cdot \alpha(b,s) \cdot F(b) \cdot F(s) \geq 0$$

- Equilibrium outcomes
  - exist (Lauermann and Nöldeke, Economics Letters 2015)
  - are feasible for the assignment problem (B, S, v, f)
  - are individually rational
  - are never pairwise stable

4. Limit outcomes

Definition 2

An outcome (x^*, u^*) is a limit outcome if there exists a sequence $(\lambda^k, x^k, u^k)_{k=1}^{\infty}$ such that

- $\lambda^k \to \infty$
- (x^k, u^k) is an equilibrium outcome for the search model with velocity parameter λ^k
- $(x^k, u^k) \to (x^*, u^*)$

- Limit outcomes exist, are feasible and individually rational.
- The question is whether they are pairwise stable, too ....

4. Limit outcomes

Preview of results

We find:

1. Limit outcomes may fail to be stable.
2. If a limit outcome is unstable, then it must feature excessive matching – frictions cause too much trade.
3. Simple sufficient conditions ensuring the stability of all limit outcomes.
4. Bounds on the efficiency loss that may arise in a limit outcome.

4. Limit outcomes

Example with unstable limit outcome

- $B = \{b_1, b_2\}$, $S = \{s_1, s_2\}$
- $f(b_1) = 9$, $f(b_2) = 2$, $f(s_1) = 9$, $f(s_2) = 1$
- $v(b,s)$ given by

	s_2	s_1
b_2	10	2
b_1	2	-6

- Unique stable outcome $(\hat{x}, \hat{u})$:

$$\hat{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\hat{u}(b_1) = 0, \quad \hat{u}(b_2) = 2, \quad \hat{u}(s_1) = 0, \quad \hat{u}(s_2) = 8$$

4. Limit outcomes

Example with unstable limit outcome

- There is (another) limit outcome given by

$$x^* = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

  and

$$u^*(b_1) = 0, \;\; u^*(b_2) = 2, \;\; u^*(s_1) = 0, \;\; u^*(s_2) = 2$$

- The unstable limit outcome is supported by a sequence of equilibria in which all matches with $v(b,s) > 0$ are consummated
- For high λ, these equilibria reflect a coordination failure: high-value agents on one side of the market are too eager to match because high-value agents on the other side of the market are also too eager too match
- This example is robust

4. Limit outcomes

Some Terminology

Let (x^*, u^*) be a limit outcome. We say that

- Type t is fully matched if $x^*(t,t) = 0$
- Type t is partially matched if $0 < x^*(t,t) < f(t)$
- Type t is unmatched if $x^*(t,t) = f(t)$
- Type pair (b,s) is a blocking pair if $u^*(b) + u^*(s) < v(b,s)$

4. Limit outcomes

Properties of limit outcomes

Lemma 1

Let (x^*, u^*) be a limit outcome in which (b, s) is a blocking pair. Then

1. *b and s are fully matched: $x^*(b,b) = x^*(s,s) = 0$*
2. *b and s obtain strictly positive payoffs:*

$$u^*(b) > 0 \quad u^*(s) > 0$$

3. *b and s do not match with any fully matched types:*

$$x^*(s',s') = 0 \Rightarrow x^*(b,s') = 0, \quad x^*(b',b') = 0 \Rightarrow x^*(b',s) = 0$$

(In particular, they do not match with each other)

4. Limit outcomes

Properties of limit outcomes

Lemma 2

Let (x^*, u^*) be a limit outcome. Then

1. *Types that are not fully matched receive a payoff of zero:*

$$x^*(t,t) > 0 \Rightarrow u^*(t) = 0$$

2. *Types that match with each other share the value of the corresponding match:*

$$x^*(b,s) > 0 \Rightarrow u^*(b) + u^*(s) = v(b,s)$$

Properties of limit outcomes

Lemma 3

Let (x^*, u^*) be a limit outcome in which (b, s) is a blocking pair. Then there exist partly matched types $b' \neq b$ and $s' \neq s$ such that b and s are fully matched with these types:

$$x^*(b, s') = f(b)$$
$$x^*(b', s) = f(s)$$

Proof (for b; same argument applies to s):

- b obtains a strictly positive payoff and must be fully matched (Lemma 1)
- ...with types that are partially matched (Lemma 1)
- Partners of b obtain payoff of zero (Lemma 2). Hence, all b-agents match with the same partner type, s' (genericity assumption on v)

4. Limit outcomes

Properties of limit outcomes

Proposition 1

Let (x^*, u^*) be an unstable limit outcome. Then there exist a stable outcome $(\hat{x}, \hat{u})$ such that

$$\hat{x}(t,t) \geq x^*(t,t)$$
$$\hat{u}(t) \geq u^*(t)$$

holds for all types t and

$$\sum_t \hat{x}(t,t) > \sum_t x^*(t,t)$$
$$V(\hat{x}) > V(x^*)$$

Proof exploits Lemma 3 to construct an auxiliary assignment problem involving only types in blocking pairs

4. Limit outcomes

Sufficient conditions for stability

Proposition 2

Every limit outcome is stable if all types have the same mass.

Proposition 3

Every limit outcome is stable if $v(b,s) > 0$ *holds for all types.*

4. Limit outcomes

Bounding the efficiency loss

Assumption 2 (Monotonicity)

The sets B and S are totally ordered and

$$v(b,s) > 0 \Rightarrow v(b',s) > v(b,s) \text{ and } v(b,s') > v(b,s)$$

for all $b' > b$ and $s' > s$.

Monotonicity ensures that all blocking types must match with the same type on the other side of the market

4. Limit outcomes

Bounding the efficiency loss

Define

- $\bar{f} = \max_t f(t)$
- $\bar{v} = \max_{(b,s)} v(b,s)$

Proposition 4

Suppose the assignment problem is monotonic. Let x^* be a limit assignment and $\hat{x}$ be an optimal assignment. Then

$$V(x^*) \geq V(\hat{x}) - 2\bar{f}\bar{v}$$

Not the best possible bound – it only exploits the structure identified in Lemma 3