Simultaneous Versus Sequential Disclosure*

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Abstract

We study an environment in which a decision maker has access to several expert advisers. The experts all have access to an identical set of facts. The decision maker's utility is increasing in the number of facts that the experts reveal. The experts have (potentially) different preferences. The game in which experts simultaneously disclose information typically has multiple equilibria. When multiple equilibria exist, the decision maker's favorite equilibrium fails to survive iterative deletion of weakly dominated strategies. We characterize the set of equilibria that survive iterative deletion of weakly dominated strategies. In a leading special case, only one outcome survives iterative deletion of weakly dominated strategies. It is the most preferred equilibrium from the perspective of the experts. We study the outcomes that can arise when the decision maker can consult the experts sequentially. We demonstrate that if the decision maker can select the order of consultations, can consult experts multiple times, and can commit to ending the consultation process, then in leading cases he can induce the same disclosure as with simultaneous disclosure. Sequential disclosure may perform worse than simultaneous disclosure from the perspective of the decision maker when it is not possible to consult experts multiple times or if commitment is not feasible.

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1 Introduction

Decision makers often take actions about things that they do not understand. When someone experiences pain, he may consult a doctor, who describes treatment options. When a car breaks down, the driver may ask a mechanic for advice. When a firm considers a product innovation, it may consult division managers before deciding a marketing strategy. Often, decision makers approach these problems by consulting just one expert. But there are situations in which it is common to request advice from more than one source. We are interested in these situations. We focus on two questions: what is the value of consulting more than one expert and how best to structure the consultations.

It is common in the literature to assume that there is an unknown state of the world. In such a model, disclosures provide information about that state. In contrast, we consider an abstract model in which experts have access to elements in a finite set. Experts choose which element to reveal. The decision maker can use everything revealed to make his decision. In our preferred interpretation, disclosures introduce new actions to the decision maker. In the case of a consultation with a doctor or a mechanic, the disclosure may be a treatment option. In the case of an executive, the disclosures may be marketing strategies or demonstrations of the valuable qualities of the new product. The disclosures may equip a department chair with arguments to persuade administrators to hire the job candidate. We view experts as individuals capable of permitting a decision maker to do something that would not be feasible without consultation. Once a doctor describes a surgical procedure, a mechanic suggests a repair, or a marketing advisor describes an ad campaign, the approach becomes available to the decision maker. Without advice, the decision maker would not have access to the idea. We assume that more disclosure benefits the decision maker (as he can ignore disclosures that he does not use), but experts may have different incentives. It is the conflict of interest between decision maker and expert that creates the possibility that the decision maker does not learn everything needed to make a decision from a single expert. Because of the possibility of incomplete disclosure, the decision maker might gain from consulting more than one expert.

Formally, we study a model in which a finite set of experts have access to an identical, partially ordered set of "facts." They play a game in which facts are disclosed. Their payoffs depend on the maximum (component wise when the set of facts is multidimensional) disclosure. We do not model the decision maker explicitly, but we assume that he makes a decision that depends on the maximum disclosure. The experts have preferences over the decision. We study two different ways of structuring the consultation process. In the first, the experts simultaneously make disclosures. In the second, the decision maker consults experts in a sequence.

An immediate result from this formulation is that competition between experts permits the decision maker to acquire the maximum amount of information when disclosure is simultaneous. The reason is straightforward: if one expert discloses everything, then it is a best response for other experts to do the same. While intuition suggests that full disclosure is possible with only two experts, we argue that the full disclosure result is too good to be true. More precisely, we show that an equilibrium refinement typically eliminates the full-disclosure outcome. The formal result is stronger than this. To get a flavor of the results, consider the case in which disclosure is one-dimensional (possible disclosures are completely ordered). In this setting, the simultaneous disclosure game typically has multiple Nash equilibrium disclosures. Moreover, the experts have the same preferences over equilibria: They prefer equilibria in which the maximum disclosure is smaller. (If an expert preferred disclosure x' to disclosure $x < x'$, then the maximum disclosure would never be x in equilibrium.) The decision maker's preference, of course, is the opposite: he prefers more disclosure to less. We show for generic payoffs that when disclosure is simultaneous, iterative deletion of weakly dominated strategies selects the experts' most preferred equilibrium. That is, the full-disclosure equilibrium is not robust if there exists another equilibrium. The paper extends this result to multi-dimensional disclosure and non-generic payoffs.

Because simultaneous disclosure need not lead to full disclosure, it is natural to ask if different ways of acquiring advice could lead to more disclosure. We study a family of sequential disclosure procedures. Here our main result is an equivalence theorem. Again, for simplicity, consider the one-dimensional case with generic preferences. We show that sequential disclosure can never do better for the decision maker than simultaneous disclosure, but that the decision maker is always able to design a sequential procedure that duplicates the outcome that survives iterative deletion of weakly dominated strategies with simultaneous disclosure. The sequential procedure requires that the decision maker have the ability to do three things: pick the order that he consults experts; commit to ending the consultation process; and have the power to return to an expert (giving her multiple opportunities to make a disclosure). This result suggests that our model does not provide a strong reason to favor simultaneous disclosure over sequential disclosure or vice versa. The paper again generalizes the finding to multi-dimensional disclosure and non-generic payoffs. The qualitative results are now less clear cut: we provide examples under which simultaneous consultation permits more disclosure than sequential consultation and others where sequential consultation allows more disclosure, but even when the set of disclosures is not linearly ordered we obtain similar bounds on the range of disclosures that can arise.

The analysis provides several other insights. Given a group of experts, we can identify when it is unnecessary to consult an expert. We can also establish a comparative-static result that confirms the intuition that adding experts must be weakly beneficial to the decision maker.

A very special case might provide useful intuition for the results. Suppose that experts have single-peaked preferences over a one-dimensional disclosure. That is, Expert i is characterized by an optimal disclosure x_i^*. Her preferences increase for $x < x_i^*$ and decrease thereafter. In this setting, it is intuitive that disclosures greater than x_i^* are weakly dominated and that the salient prediction for the simultaneous disclosure game is that the decision maker will learn the maximum of the x_i^*. It is also apparent that the decision maker need only consult a single expert – the one with the maximum x_i^*. When preferences are not single peaked, the analysis is more complicated. By constructing a sequential procedure with repeated consultations, the decision maker can secure a disclosure of one expert's global maximum and then use this knowledge to induce further disclosures from other experts.

We organize the remainder of the paper as follows. Section 2 describes some of the papers related to ours. Section 3 describes the basic model. Section 4 contains the analysis of the simultaneous disclosure game. Section 5 contains the analysis of the sequential disclosure game. Section 6 compares simultaneous to sequential disclosure. Section 7 contains concluding remarks. The appendix contains proofs that are not in the main text.

2 Related Literature

There is an extensive literature that discusses whether competition between information sources leads to better decisions. Gentzkow and Kamenica (2017a) and Gentzkow and Kamenica (2017b) study a model in which experts simultaneously choose how much to communicate to a decision maker in a Bayesian Persuasion framework. In these models, the decision maker wants to know the value of the state of the world and the strategies of experts are arbitrary signals (joint probability distributions on the state and message received by the decision maker). Gentzkow and Kamenica (2017b) shows that adding an agent need not increase the amount of information revelation, but provides a condition under which increasing the number of experts increases the amount of information revealed. Although we do not interpret our model in this way, one can view our model as an example of the Bayesian Persuasion problems studied by Gentzkow and Kamenica with restrictions on the kind of signals available to the experts. Additional experts are always valuable in the sense that the minimal equilibrium disclosure is increasing in the number of experts. Gentzkow and Kamenica do not focus on equilibrium selection, but they note the existence of multiple equilibria and the tendency of experts to prefer less disclosure. Li and Norman (2018b) studies a sequential version of the Gentzkow and Kamenica model. They provide an existence and partial characterization result. They show that sequential persuasion results in no more informative equilibria than simultaneous persuasion. Li and Norman (2018a) also notes that the order of disclosure matters, pointing out it is possible that adding an expert into a sequence may decrease the amount of information disclosure. In addition to the different interpretation of the nature of the disclosure game, our analysis contributes an equilibrium refinement of the simultaneous game and studies the optimal order of consultation in the sequential game. We describe the formal condition between our model and Bayesian Persuasion in Section 3.

Inés Moreno de Barreda suggested another interpretation of our model. Her proposal was to interpret an Expert's strategy as permission to undertake certain activities. (If Expert i "announces" x_i, then the decision maker can pursue any activity less than or equal to x_i.) Perhaps a division manager must secure parts from different subdivisions or a teenager needs permission from just one parent in order to stay out to a certain hour or travel to a certain location.

In a different context, Auster and Pavoni (2019) model unawareness of strategic options in a way that is similar to our leading interpretation of the model.

Krishna and Morgan (2001a) and Krishna and Morgan (2001b) study competition in disclosure in a cheap-talk setting. When experts report simultaneously, they construct a fully revealing equilibrium, but they show that full disclosure need not be an equilibrium when experts move sequentially. Milgrom and Roberts (1986) study the issue in a model of verifiable information. In this setting, full disclosure is an equilibrium with only a single expert in leading cases (Grossman (1981) and Milgrom (1981)).

We know of three papers that compare simultaneous to sequential interactions in different contexts. Dekel and Piccione (2000) compare simultaneous to sequential voting institutions. There are a finite number of voters and two options. Voters can either vote for or against the status quo. Voters do not know their valuations, but receive private signals. Dekel and Piccone compare the equilibria of games in which voters cast votes simultaneously to those in which votes are sequential. They show that a symmetric informative equilibrium of the simultaneous game is an equilibrium to any sequential game. Weaker results hold for asymmetric equilibria.[1] Although this paper researches a conclusion that is similar to ours, we do not see a formal connection between the analyses. Dekel and Piccone's model focuses on aggregation of preferences, while our setting involves disclosure. Our equivalence result requires an equilibrium refinement and commitment power in the design of sequential disclosure mechanisms. Schummer and Velez (2018) identify conditions under which social choice functions that can be implemented in truthful strategies when players move simultaneously cannot be truthfully implemented when players move sequentially. The context is quite different from our paper, but it suggests environments in which sequential procedures will perform less well than simultaneous ones.

Doval and Ely (2019) characterize all equilibria that can arise from some information structure and some extensive form (for a fixed set of players and preferences over final outcomes). In their construction, they introduce a "canonical extensive form" that is sufficient to generate any equilibrium. In a canonical extensive form, each player moves at most once. Our construction requires that an individual player may move more than once. The reason for this difference is that Doval and Ely's construction requires a partial commitment assumption that requires that once a player has made an action choice, that player can have no other payoff relevant moves. This assumption does not hold in our model.

Glazer and Rubinstein (1996) show that given a normal game form (strategies and players) one can construct an extensive game form that is equivalent to the normal game form and for any specification of preferences, the normal-form game is dominance solvable if and only if the extensive-form game is solvable by backward induction. Glazer and Rubinstein argue that the transformation makes it easier to carry out the process of removing dominated strategies, suggesting that the extensive-form game is easier to play. Our construction associates with a dominance-solvable normal-form game an extensive-form game using a communication protocol, but it is typically not the case that the extensive-form game is equivalent to the normal-form game.

3 Underlying Strategic Environment

There is a finite set of players. I denotes the player set.[2] We assume that there is a common set of strategies X available to each player. We assume that there is a binary relation $>$ that is transitive and irreflexive. For $x, x' \in X$, we write $x \geq y$ if $x > y$ or

[1] Dekel and Piccione (2014) study a voting model in which the timing of votes is a strategic choice.

[2] In an abuse of notation, we also let I denote the cardinality of the player set.

$x = y$. Relations $<$, $\leq$, $\not>$, $\not\leq$, $\not\geq$, are defined in the standard way. We assume that $(X, >)$ so that $\min\{z \in X : z \geq x, x'\}$ and $\max\{z \in X : z \leq x, x'\}$ are defined for all x and x'. We let $x \vee x' = \min\{z \in X : z \geq x, x'\}$ and $x \wedge x' = \max\{z \in X : z \leq x, x'\}$.

For $y = (y_1, \ldots, y_l)$, $y_i \in X$ let $M(y) = y_1 \vee \cdots \vee y_l$. Each expert i has a utility function $\tilde{u}_i : X \to \mathbb{R}$. We assume that there exist $u_i : X \to \mathbb{R}$ such that $\tilde{u}_i(x) \equiv u_i(M(x))$. We call elements of X disclosures. We denote the minimal element of X by $\underline{x}$.

We assume each $u_i(\cdot)$ is quasi-supermodular.[3] Quasi-supermodularity is a complementarity assumption that implies, roughly, that increasing the disclosure in one dimension makes disclosure in another dimension more attractive. It will hold if utility is separable across components, but it will fail if facts in one dimension substitute for facts in another dimension (and experts strictly prefer intermediate disclosures).

We view this setting as a reduced form for disclosure games. The player set consists of a group of experts. There are n issues. For each issue, there is an ordered list of "facts." Each of the agents has access to these facts. A strategy for a player describes the facts that the expert discloses to a decision maker. The decision maker aggregates the disclosures $y_1, \ldots, y_I$ and can make a decision based on all of the facts presented, $M(y_1, \ldots, y_I)$. We do not model the choices of the decision maker explicitly, but we have in mind a situation in which the decision maker selects an action given available facts and that the optimal action rule is a function of the join of all of the disclosures. Keeping with this interpretation, we assume that the decision maker's utility is increasing in what is disclosed.

One version of our interpretation is that there is an underlying state of the world and experts provide the decision maker with "experiments" – procedures that produce for each state of the world a probability distribution over a set of signals observable by the decision maker. The decision maker then makes a decision based on the signals he observes (and knowledge of the experiments and the prior distribution on the state of the world). This interpretation is consistent with the model of competition in persuasion in Gentzkow and Kamenica (2017b). Let us describe the connection in somewhat more detail. We restrict attention to finite environments. In any Bayesian Persuasion problem, there is a given state space, Θ. We create a new state space $\Theta^* \equiv \Theta \times T$ where $(\theta, t) \in \Theta^*$, t is independently and uniformly distributed on a finite set T. A partition of Θ^* is an experiment (in the sense that observing an element of the partition generates a posterior distribution on Θ). Provided that we allow only finitely many experiments, the Bayesian Persuasion model translates into our framework: We must only interpret strategy sets as partitions. The strategy set has a lattice structure: If x and x' are two partitions, then $x \vee x'$ is the common refinement ($\{Q = P \cap P'$ for $P \in x, P' \in x'\}$) and $x \wedge x'$ is the finest coarsening (a partition z such that for all $P \in x$ ($P' \in x'$) there exists $Q \in z$ ($Q' \in z$) such that $P \subset Q$ ($P' \subset Q'$) and there is no finer partition with this property). In this way, we can interpret any finite Bayesian Persuasion problem as being captured by our model. In making the transformation, we should emphasize that every agent has access to the same strategy set and that some of our results rely on quasi-supermodularity. It is straightforward to give conditions that guarantee quasi-supermodularity in simple Bayesian Persuasion problems, but in general the condition is

[3] The function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$ and quasi-supermodular if $f(x) \geq (>) f(x \wedge y)$ implies $f(x \vee y) \geq (>) f(y)$.

restrictive. Furthermore, the upper bound for disclosures that we provide in Proposition 1 depends on the assumption that X is a cartesian product. The lattice structure derived from identifying X with (finite) partitions and $\vee$ with refinement need not have this structure. Consequently our full characterization does not apply.[4]

We prefer the interpretation that the experts provide tools for solving problems that the decision maker could not access without the experts' disclosures.

4 Simultaneous Disclosure

In this section, we study the game in which experts simultaneously select an element in X. If $x = (x_1, \ldots, x_I)$ is the profile of disclosures, then Expert i's payoff is $\tilde{u}_i(x)$.

A disclosure $\pi = (\pi_1, \ldots, \pi_I)$ such that $u_i(\pi) \geq u_i(\pi')$ for all $\pi' > \pi$ and all i is a Nash equilibrium disclosure. For any equilibrium disclosure π, a strategy profile x that satisfies $x_i \leq \pi$ and at least two $x_j = \pi$ is a Nash Equilibrium. Full disclosure arises when the maximum disclosure is equal to the maximum element in X. Full disclosure is always an equilibrium disclosure. It can be supported by a strategy profile in which at least two players reveal the maximal element of X. Typically there are other Nash Equilibria. The pure-strategy Nash Equilibria are Pareto ranked when information structures are linearly ordered (if x^* and x^{**} are equilibrium disclosures and $x^{**} > x^*$ then all experts prefer x^* to x^{**}). The equilibria are not necessarily Pareto ranked if information structures are partially ordered, but even in this case, if x^* and x^{**} are both Nash Equilibria and $M(x^*) \leq M(x^{**})$, then $\tilde{u}_i(x^*) \geq \tilde{u}_i(x^{**})$ for all i.

The possibility of multiple equilibria leads us to consider a more restrictive solution concept.

Definition 1. *Given subsets $X_i' \subset X$, with $X' = \prod_{i \in I} X_i'$, Player i's strategy $x_i^* \in X_i'$ is a best response to $x_{-i} \in X_{-i}'$ relative to X_i if $\tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i, x_{-i})$ for all $x_i \in X_i$. Player i's strategy $x_i \in X_i'$ is weakly dominated relative to X' if there exists $z_i \in X_i'$ such that $\tilde{u}_i(x_i, x_{-i}) \leq \tilde{u}_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}'$, with strict inequality for at least one $x_{-i} \in X_{-i}'$.*

Definition 2. *The set $S = S_1 \times \cdots \times S_I \subset X$ survives iterated deletion of weakly dominated strategies (IDWDS) if for $m = 0, 1, 2, \ldots,$ there are sets $S^m = S_1^m \times \cdots \times S_I^m$, such that $S^0 = X$, $S^m \subset S^{m-1}$ for $m > 0$; S_i^m is obtained by (possibly) removing strategies in S_i^{m-1} that are weakly dominated relative to S^{m-1}; $S^m = S^{m-1}$ if and only if for each i no strategy in S_i^{m-1} is weakly dominated relative to S^{m-1}; and $S_i = \bigcap_{m=1}^{\infty} S_i^m$ for each i.*[5]

For finite games, it must be the case that there exists an m such that $S^r = S^m$ for all $r > m$. There are typically many different procedures that are consistent with Definition 2. These procedures may lead to different sets that survive the process. We

[4] The technical problem is that the lattice induced by partitions may fail to be distributive ($x \vee (y \wedge z)$ need not equal $(x \vee y) \wedge (x \vee z)$).

[5] Our notation follows these rules: superscripts denote steps in an iterative process; subscripts denotes players; arguments are components.

discuss properties that are common to all sets that survive and give conditions under which all sets that survive lead to the same maximum disclosure.

IDWDS is a powerful concept that makes strong demands on the rationality of agents. It is also delicate – the order of deletion matters and it is sometimes poorly behaved in games with continuous strategy spaces.[6] Nevertheless, this concept appears appropriate in contexts such as ours where an individual agent's decision is relevant to her own payoff in a limited number of circumstances. Just as in voting models one wants to condition behavior on the event that a voter is pivotal, in the disclosure model, one wants to focus attention on circumstances when an expert's disclosure will alter the total amount of information available to the decision maker. Weak dominance arguments capture these strategic circumstances.

We analyze the implications of applying iterated deletion of weakly dominated strategies. Sobel (2019) introduces a class of supermodular games called WID-supermodular games and describes general properties of strategies that survive the process of iteratively deleting weakly dominated strategies in these games. He shows that if $X \subset \mathbb{R}$, then the simultaneous disclosure game is a WID-supermodular game. It is straightforward to show that when $u_i(\cdot)$ is supermodular, the simultaneous disclosure game is a WID-supermodular game even when $\geq$ is not complete.

Quasi-supermodularity imposes structure on the set of Nash equilibria. We state some useful properties. Topkis (2011, Theorem 2.72) reports Fact 1.

Fact 1. *For any sublattice $X' \subset X$, $\arg\max_{z \in X'} u_i(z)$ is a sublattice of X.*

Hence the set of best replies to any pure strategy forms a sublattice and the smallest best response exists.

Lemma 1. *If π' and π'' are equilibrium disclosures, then $\pi' \wedge \pi''$ is an equilibrium disclosure.*

Lemma 1 implies that there is a minimum equilibrium disclosure. We have observed that maximal disclosure is an equilibrium, but now we know that a minimal equilibrium disclosure exists. We next show that every disclosure that survives iterated deletion of weakly dominated strategies, whether it is an equilibrium or not, is greater than or equal to this disclosure. Before we describe the procedure, we first define two quantities π^* and $\tilde{\pi}^*$.

Definition 3. *The smallest strictly preferred equilibrium disclosure is*

$$\pi^* = \min\{\pi : u_i(\pi) > u_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

Definition 4. *The smallest equilibrium disclosure is*

$$\tilde{\pi}^* = \min\{\pi : u_i(\pi) \geq u_i(x_i) \text{ for all } x_i > \pi \text{ and all } i\}.$$

[6] In particular, in large games there is no guarantee that there exists a Nash equilibrium in strategies that are not weakly dominated. It is for this reason that we limit attention to finite strategy spaces.

Note that these disclosures are Pareto efficient (from perspective of the experts) in the set of Nash equilibrium payoffs.

It is immediate that $\tilde{\pi}^*$ is well defined and the smallest Nash equilibrium disclosure. That π^* is well defined is a straightforward consequence of the definition. Clearly, $\pi^* \geq \tilde{\pi}^*$. Equality will hold if $u_i(\cdot)$ is one-to-one for each player. Any strategy profile x that satisfies $x_i \leq \pi$ and at least two $x_j = \pi$ is a Nash Equilibrium for $\pi = \pi^*$ and $\tilde{\pi}^*$. We will show that $\tilde{\pi}^*$ is a lower bound of disclosures, i.e., all disclosures that survive IDWDS are greater than or equal to $\tilde{\pi}^*$.

We need a bit more terminology and notation to state our main result.

Definition 5. *Let $\bar{x}(k)$ be the largest (feasible) disclosure in dimension k. Let $\bar{x}(-k)$ be the collection of largest (feasible) disclosures in dimensions other than k.*

Definition 6. *The bounding disclosure is*

$$\bar{\pi}^* = \min\{\pi : u_i(\pi(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k)) \text{ for all } x_i(k) > \pi(k), \text{all dimensions } k, \text{ and all } i\}.$$

In the appendix we show that there is a minimum in the definition exists (Claim 1). Denoting this value by $\bar{\pi}^*$ we show that $\bar{\pi}^*$ is an equilibrium and $\bar{\pi}^* \geq \pi^*$. If X is one dimensional, then π^* is the smallest bounding disclosure. It is possible that $\pi^* < \bar{\pi}^*$ (see Example 6).

Example 6 also provides insight into why the bounding disclosure may be strictly greater than π^* when X is not completely ordered. Each expert must decide if she is revealing "too much" dimension-by-dimension. The test imposed in the definition of bounding closure requires that when an expert considers reducing a disclosure in dimension k, she assumes full disclosure in the other dimensions. It is possible to refine the definition of bounding disclosure (to obtain a weakly lower bound) by replacing $\bar{x}$ by the upper bound of strategies that survive deletion of weakly dominated strategies. We omit this discussion because it adds complexity without much insight.

We can now state the main result of this section.

Proposition 1. *If x is a strategy profile that survives IDWDS, then $M(x) \in [\tilde{\pi}^*, \bar{\pi}^*]$. If x is a Nash equilibrium strategy profile that survives IDWDS, then $\tilde{u}_i(x) \geq u_i(\bar{\pi}^*)$ for all i.*

The proposition bounds the set of disclosures that survive iterative deletion of weakly dominated strategies. The lower bound is the lower bound of the set of Nash equilibria. The upper bound is typically lower than full disclosure. We provide examples to demonstrate that the bounds need not survive IDWDS, but we are able to describe conditions when the bounds are tight.

To prove the proposition, we first show that there is always a disclosure less than or equal to π^* that survives IDWDS. This observation follows because if $x_j \leq \pi^*$ for all $j \neq i$, then Expert i must have an undominated best reply to x_{-i} that is less than or equal to π^*. Next we show that disclosures strictly lower than $\tilde{\pi}^*$ must eventually be eliminated. This argument uses the definition of $\tilde{\pi}^*$ and in particular, the fact that for any $\tilde{\pi} \ngeq \tilde{\pi}^*$, there must exist an expert i such that $u_i(x_i) > u_i(\tilde{\pi})$. Finally, we show how to delete strategies that are not less than or equal to $\bar{\pi}^*$. Proving this is more involved.

The argument involves constructing a strategy that dominates the smallest remaining strategy that is not below $\bar{\pi}^*$. The appendix contains the details.

Sobel (2019, Proposition 3) proves this result when $\geq$ is complete.

Corollary 1. *If $\bar{\pi}^* = \tilde{\pi}^*$, then for all x that survives IDWDS, $M(x) = \pi^* = \bar{\pi}^* = \tilde{\pi}^*$.*

Corollary 1 follows directly from Proposition 1.

Remark 1. *When X is one dimensional, the disclosure that survives is unique if $u_i(\cdot)$ is one-to-one for all i (so that $\pi^* = \tilde{\pi}^*$).*

As the examples in the next section demonstrate, we cannot generally say more than that equilibrium disclosures must be in the interval $[\tilde{\pi}^*, \bar{\pi}^*]$. That is, the bounds are not attained in every game.

4.1 Examples

In this subsection we present examples that demonstrate that we cannot strengthen the conclusion of Proposition 1 and that, in general, the order of deleting strategies matter. In this case Proposition 1 identifies a unique payoff that survives IDWDS when payoff functions are generic and X is one dimensional. Hence the pathologies when disclosures are ordered are all due to ties in payoff functions. We cannot guarantee that disclosure π^* or disclosure $\tilde{\pi}^*$ will survive IDWDS nor can we guarantee that all payoffs that survive IDWDS are greater than or equal to $u_i(\pi^*)$.

Example 1. *Consider the following game:*

	None	*Some*	*All*
None	$2, 0$	$1, 2$	$1, 1$
Some	$1, 2$	$1, 2$	$1, 1$
All	$1, 1$	$1, 1$	$1, 1$

We have $\pi^* = All$ and $\tilde{\pi}^* = Some$. Discarding the bottom two strategies of Expert 1 (the row player) initially leaves $(None, Some)$; discarding Expert 2's None and All, leaves $(any, Some)$ (so either Some or All is disclosed). You cannot delete the outcome $(None, Some)$. So the set of disclosures that survive IDWDS always includes $\tilde{\pi}^*$, but may or may not include π^*. Although Proposition 8 guarantees that π^* is a disclosure that survives IDWDS for some order of deletion, this example demonstrates that we cannot guarantee that π^* survives independent of the deletion order.

Example 2. *Consider the following game:*

	None	*Some*	*All*
None	$1, 1$	$-1, 0$	$1, 0$
Some	$-1, 0$	$-1, 0$	$1, 0$
All	$1, 0$	$1, 0$	$1, 0$

We have $\pi^* = All$ and $\tilde{\pi}^* = None$. Discarding Expert 2's Some and All, leads to $(None, None)$ and $(All, None)$. Discarding Expert 1's None and Some, leads to (All, Any). You cannot delete Expert 1's All strategy. Here the set of disclosures that survive IDWDS always includes π^*, but may or may not include $\tilde{\pi}^*$.

Taken together, the examples show that you need not select π^* or $\tilde{\pi}^*$. The examples are consistent with the observation that you will select one or the other and that there is a way of deleting weakly dominated strategies that will select both. The second claim is not true, however.

Example 3. *Consider the following game:*

	None	*Some*	*All*
None	$0, 0$	$0, -1$	$-1, -1$
Some	$0, -1$	$0, -1$	$-1, -1$
All	$-1, -1$	$-1, -1$	$-1, -1$

We have $\pi^* = All$ and $\tilde{\pi}^* = None$. (Some, None) and (None, None) are equilibria that survive IDWDS, but it is weakly dominated to disclose All. Consequently $\tilde{\pi}^*$ is an equilibrium disclosure that survives IDWDS; π^* is not an equilibrium disclosure that survives IDWDS; and there is an equilibrium disclosure that survives IDWDS strictly between $\tilde{\pi}^*$ and π^*.

It is also possible to construct an example in which $\tilde{\pi}^*$ is not an equilibrium disclosure that survives IDWDS.

Example 4. *Consider the following game:*

	None	*Some*	*All*
None	$2, 1$	$0, 1$	$2, 0$
Some	$0, 1$	$0, 1$	$2, 0$
All	$2, 0$	$2, 0$	$2, 0$

We have $\pi^* = All$ and $\tilde{\pi}^* = None$. The only equilibrium disclosure that survives IDWDS is π^*.

IDWDS need not bound the utility of the experts.

Example 5. *Consider the following game:*

	None	*Some*	*All*
None	$1, 0$	$-1, 0$	$0, 0$
Some	$-1, 0$	$-1, 0$	$0, 0$
All	$0, 0$	$0, 0$	$0, 0$

In this example, $\pi^* = All$ and $\tilde{\pi}^* = None$. Expert 1's "Some" strategy is weakly dominated, but all other strategies survive. Consequently there exists an outcome that survives IDWDS in which Expert 1's payoff is less than $u_1(\pi^*)$. This outcome, $(None, Some)$, is not an equilibrium.

Now we turn to a situation in which the strategies are not completely ordered.

Example 6. *Consider the following game:*

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$2,2$	$1,-1$	$-1,1$	$0,0$
$(1,0)$	$1,-1$	$1,-1$	$0,0$	$0,0$
$(0,1)$	$-1,1$	$0,0$	$-1,1$	$0,0$
$(1,1)$	$0,0$	$0,0$	$0,0$	$0,0$

The two players' preferences are: $\{0,0\} \succ_1 \{1,0\} \succ_1 \{1,1\} \succ_1 \{0,1\}$ *and* $\{0,0\} \succ_2 \{0,1\} \succ_2 \{1,1\} \succ_2 \{1,0\}$*. There are* 2 *pure-strategy equilibrium disclosures:* $(0,0),(1,1)$*. It is dominated for Expert* 1 *to disclose* 1 *on the second dimension and for Expert* 2 *to disclose* 1 *on the first dimension. So discard* $(1,1),(0,1)$ *for Expert* 1 *and* $(1,1),(1,0)$ *for Expert* 2*. No other strategy can be deleted. Every disclosure survives IDWDS. In this example* $\pi^* = \tilde{\pi}^* = (0,0)$*, but* $\bar{\pi}^* > \pi^*$ *and there are disclosures that survive IDWDS that exceed* π^**.*

Example 6 illustrates how full disclosure may survive iterative deletion of weakly dominated strategies in the simultaneous game even when the strategy of full disclosure is weakly dominated, which cannot happen in the one-dimensional case. In the example, $(1,0)$ is an unattractive disclosure from the perspective of Expert 2. If this disclosure is a possibility, then Expert 2 will have a justification for making a disclosure in the second dimension. Similarly, Expert 1 would not delete her strategy $(1,0)$ because she prefers that the decision maker have full information than information only about the second dimension. Although no disclosure remains an equilibrium that survives IDWDS, the full disclosure outcome is more robust than it is in the one-dimensional case.

5 Sequential Disclosure

Full disclosure does not always survive iterative deletion of weakly dominated strategies when Experts disclose simultaneously. This leaves open the question of whether the decision maker could do better by consulting the experts in a different way.

What will be generated by an alternative arrangement where each expert sequentially discloses information and observes prior disclosures when making their choice? This section gives conditions under which the outcomes in these two environments are essentially the same.

5.1 Preliminaries

Let $H_0 = \emptyset$, $H_t = X^t$, where X is the (finite) set of possible disclosures.[7] Let $H = \bigcup_{t=0}^{\infty} H_t$ be the set of histories. If $h_t = (h_t^1, \ldots, h_t^t) \in H_t$ and $h_{t'} = (h_{t'}^1, \ldots, h_{t'}^{t'}) \in$

[7] Histories should also include the identity of the expert who reveals, but not just what is revealed. The extra generality does not affect the analysis so we use a simpler definition of history.

$H_{t'}$, then $h_t h_{t'} \in H_{t+t'}$ is the history obtained by the natural concatenation: $h_t h_{t'} = (h^1, \ldots, h^t, h^{t+1}, \ldots, h^{t+t'})$ where

$$h^m = \begin{cases} h_t^m & \text{if } 1 \le m \le t \\ h_{t'}^{m-t} & \text{if } t < m \le t + t'. \end{cases}$$

Definition 7. *A finite sequential disclosure protocol is a mapping $P : H \to \{0, 1, \ldots, I\}$ such that for all $h, h_t \in H$,*

$$P(h_t) = 0 \Longrightarrow P(h_t h) = 0 \tag{1}$$

and there exists T such that $P(h_T) = 0$ for all $h_T \in H_T$.

The decision maker observes a history of disclosures, h_t. He then decides whether to stop the process ($P(h_t) = 0$) or to ask Expert i to disclose ($P(h_t) = i$). Condition (1) means that once the decision maker stops the process, he cannot restart it. We limit attention (without loss of generality) to protocols that end after a finite number of periods.

A sequential disclosure protocol induces a game in which the players are the experts. Player i's strategy specifies a disclosure as a function of h_t for each h_t such that $P(h_t) = i$. Given a history of length t, $h_t = (h_t^1, \ldots, h_t^t)$, let $\mu(h_t) = \max\{h_t^1, \ldots, h_t^t\}$, where the maximum is taken component wise. A strategy profile $s = (s_1, \ldots, s_I)$ determines disclosures $d_t^*(s)$ and histories $h_t^*(s)$ for $t = 1, \ldots, T$ where $h_1^*(s) = d_1^*(s) = s_{P(\emptyset)}(\emptyset)$, $d_2^*(s) = s_{P(h_1^*(s))}(h_1^*(s))$, $h_2^*(s) = h_1^*(s) d_2^*(s)$, and, in general, $d_k^*(s) = s_{P(h_{k-1}^*(s))}(h_{k-1}^*(s))$, $h_k^*(s) = h_{k-1}^*(s) d_k^*(s)$ with the convention that $s_0(h) = \mu(h)$. Expert i's payoff as a function of the strategy profile is $\tilde{u}_i(s) = u_i(\mu(h_T^*(s)))$. We say that a disclosure π is generated by a sequential disclosure protocol if the induced game has a strategy profile that survives IDWDS in which π is disclosed. A disclosure π is uniquely generated by a sequential disclosure protocol if π is the only disclosure generated by the protocol.

5.2 Example

The definition of sequential protocol permits the decision maker to do three things: vary the order in which he consults experts; return to experts more than once; and commit to ending the consultation process. The following example demonstrates why these three features are important and gives some insight into the general construction.

Example 7. *There are five information structures, $1, 2, 3, 4, 5$ with higher numbers representing more information, and two experts. Expert 1 has strict preferences: $2 \succ 4 \succ 1 \succ 5 \succ 3$. Expert 2 has strict preferences: $1 \succ 3 \succ 2 \succ 5 \succ 4$. Here $\bar{\pi}^* = \pi^* = \tilde{\pi}^* = 5$. So the unique outcome that survives IDWDS in the simultaneous game is full disclosure. There are four possible disclosure sequences without returning to an expert: consulting exactly one expert, or consulting both in either order. Without commitment, the possible disclosures are:*

Sequence	*Disclosure*
Expert 1	2
Expert 2	1
Expert 1, then 2	1
Expert 2, then 1	2

It is straightforward to confirm that returning to experts will not lead to more disclosure. Hence, without commitment, sequential disclosure need not lead to disclosure π^*.

If the decision maker can consult many experts, but can never consult an expert more than once, then commitment ability cannot improve upon the protocols in which only one expert is consulted (in these cases, the decision maker commits to stop after the first disclosure independent of the disclosure itself). If he starts with Expert 1 and sometimes asks Expert 2, then more disclosure is possible with commitment. If the decision maker stops after 1, 2, *or* 3, *Expert 1 will disclose* 2, *which will be the final disclosure. If the decision maker stops after* 4 *or* 5, *the disclosure will be* 4. *This means that if the decision maker consults Expert 1 first, then he would do best by committing not to consult Expert 2 if Expert 1 discloses at least* 4. *Alternatively, the decision maker can consult Expert 2 first. The disclosure generated will be* 1 *if the decision maker stops after disclosure* 1; 2 *if the decision maker stops after* 2, 4 *or* 5, *and* 3 *if the decision maker stops at* 3. *This means that the decision maker can obtain disclosure* 3 *by consulting first Expert 2, then Expert 1 (with commitment). We conclude that the best the decision maker can do with commitment but without returning to experts is disclosure* 4. *Hence commitment increases the disclosure, but does not generate full disclosure.*

Proposition 2, which we state and prove in Section 5.3 guarantees that there is a sequential disclosure protocol that generates disclosure $\tilde{\pi}^*(= \pi^* = \bar{\pi}^*)$. *The protocol involves asking Expert 1, then Expert 2, and then going back to Expert 1, with the commitment to stop if Expert 2 discloses* 3.

Adding an additional expert is quite beneficial to the decision maker in the example. Full disclosure is possible even though it is ranked fourth out of five disclosures for both experts.

The example illustrates the importance of the three features of sequential protocols. The order of consultation matters. Consulting Expert 2 is better for the decision maker than consulting Expert 1 first because Expert 1 is certain to disclose more than 1 if she moves last. Commitment is important because it motivates an expert to make a partial disclosure without fear that a future expert will disclose more. Finally, returning to an expert may be valuable because an expert's favorite disclosure depends on what has already been disclosed. In the example, full disclosure is unattractive to Expert 2 ex ante, but once 4 has been disclosed, it is her favorite option. What complicates the construction is the possibility that an expert will want to disclose more in response to a partial disclosure from another expert. Well designed sequential protocols permit the decision maker to leverage conflicts of interest between experts to obtain greater disclosure.

5.3 The Canonical Sequential Disclosure Protocol

This subsection describes what is possible for the decision maker using a particular sequential protocol. The next subsection will explain the importance and limits of this protocol.

Imagine that the decision maker commits to consulting only Expert i. Expert i would disclose an x that maximizes $u_i(x)$. Provided that $\underline{x} < \tilde{\pi}^*$, this disclosure is greater than $\underline{x}$ for at least one i. Consequently, the decision maker can guarantee a positive disclosure. Call this disclosure π^1. Now imagine the decision maker reaches a stage in which $x < \tilde{\pi}^*$ has been disclosed. By the definition of $\tilde{\pi}^*$ this means that some expert can gain strictly by disclosing more (if she were guaranteed that the decision maker would consult no further experts). Hence if the decision maker can guarantee the disclosure x, then by picking the correct expert to consult next, he would be able to guarantee further disclosure.

We construct a protocol that builds on this insight. We define a finite, increasing sequence of disclosures that satisfies

$$\underline{x} = \pi^0 < \pi^1 < \cdots < \pi^{\bar{m}} = \tilde{\pi}^* \tag{2}$$

such that the decision maker can guarantee a disclosure of at least π^m by consulting m times. The basic idea is that as long as $\pi < \tilde{\pi}^*$, the decision maker can leverage the ability to guarantee disclosure π to induce an expert who prefers to disclose more than π to make a further disclosure. We observe that this task would be straightforward if experts had single-peaked preferences. In that case, the decision maker need only consult a single expert (the one with the greatest peak).

Describing the protocol requires introducing some notation.

Let $\bar{v}_i(x) = \min \arg\max\{u_i(v) : v \geq x\}$. The quantity $\bar{v}_i(x)$ is the smallest disclosure that maximizes Expert i's utility given that x has already been disclosed. (If there is more than one disclosure that maximizes Sender i's utility, then $\bar{v}_i(x)$ is the smallest one.) Let $\pi^0 = \underline{x}$; and for $m > 0$,

$$\underline{i}(m) = \min\{i : \bar{v}_i(\pi^{m-1}) = \max_j\{\bar{v}_j(\pi^{m-1})\}\}$$

and $\pi^m = \bar{v}_{\underline{i}(m)}(\pi^{m-1})$. Given that $\bar{v}_i(\cdot)$ is well defined, so are $\underline{i}(m)$ and π^m for all m. $\underline{i}(m)$ is the expert whose smallest maximizing disclosure is the greatest given that π^{m-1} has already been disclosed. In case of ties we select the expert with the smallest index among those with the largest $\bar{v}_j(\pi^{m-1})$.

Note that $\underline{i}(m) \neq \underline{i}(m-1)$, but it is possible to have $\underline{i}(m) = \underline{i}(m')$ for $m > m'+1$. That is, this disclosure mechanism may require asking the same expert to disclose more than once. Finally we define π^m to be the smallest disclosure of Expert $\underline{i}(m)$ that maximizes her payoff given that π^{m-1} has already been disclosed.

It follows from the definition of $\tilde{\pi}^*$ that if $x \not\geq \tilde{\pi}^*$, then $\bar{v}_{\underline{i}}(x) > x$ for all x. Hence, there is a finite $\bar{m}$ such that $\pi^m \not\geq \tilde{\pi}^*$ for $m < \bar{m}$ and $\pi^{\bar{m}} \geq \tilde{\pi}^*$. The next result refines this observation.

Lemma 2. $\pi^m < \tilde{\pi}^*$ *for* $m < \bar{m}$ *and* $\pi^{\bar{m}} = \tilde{\pi}^*$.

Lemma 2 implies that (2) holds.

We now can define the disclosure protocol.

Definition 8 (Canonical Disclosure Protocol)**.** *In the Canonical Disclosure Protocol, for any $h_t \in H_t$,*

$$P(h_t) = \begin{cases} 0 & \text{if } \mu(h_t) \geq \pi^{\bar{m}-t} \\ \underline{i}(\bar{m}-t) & \text{otherwise.} \end{cases}$$

In the canonical disclosure protocol the decision maker first asks Expert $\underline{i}(\bar{m})$; if this expert discloses at least $\pi^{\bar{m}-1}$, then the decision maker stops. Otherwise, he moves to Expert $\underline{i}(\bar{m}-1)$. The process continues. If the disclosure after the r^{th} report is at least $\pi^{\bar{m}-r}$, then the decision maker stops (and requests no further reports). Otherwise, the decision maker requests a report from Expert $\underline{i}(\bar{m}-r)$.

Proposition 2. *If π is a disclosure that survives IDWDS in the game determined by the canonical disclosure protocol, then $\pi \geq \tilde{\pi}^*$.*

Proposition 2 provides a lower bound to what the decision maker can obtain from the canonical sequential disclosure protocol.

Lemma 3. *If $\tilde{\pi}^* \leq \pi$ and $\pi \not\leq \pi^*$, then $u_i(\pi) < u_i(\tilde{\pi}^*)$.*

The next result provides an upper bound for this sequential mechanism.

Proposition 3. *If π is generated by the canonical protocol, then $\pi \leq \pi^*$.*

Corollary 2. *If $\tilde{\pi}^* = \pi^*$, then there exists a sequential disclosure protocol that uniquely generates the disclosure π^*.*

In particular, if the experts all have strict preferences, then the canonical sequence uniquely generates the disclosure π^*. Corollary 2 is an immediate consequence of Propositions 2 and 3.

We next consider situations in which $\tilde{\pi}^* < \pi^*$. Proposition 2 guarantees that at least $\tilde{\pi}^*$ is disclosed. There may be no protocol that guarantees disclosure π^*. A simple example demonstrates this fact. Assume there is a single expert and two disclosures (high and low), and the expert is indifferent between the two. Every protocol will have an outcome with high disclosure and an outcome with low disclosure.

5.4 General Bounds on Sequential Consultation

We have focused on the canonical disclosure protocol in the previous subsection. This protocol is sufficient to generate the highest disclosure feasible when X is one dimension. In general, the results are less clear cut. This subsection explains what is and is not possible using general protocols.

Proposition 4. *For all $x > \pi^*$, there exists no sequential disclosure protocol that generates the disclosure x in a pure-strategy, subgame-perfect equilibrium.*

Proposition 4 follows from backward induction. After each history that discloses no more than π^*, it is never a best response to disclose more than π^*. So if the first expert anticipates that the final disclosure will be more than π^*, she can do strictly better by disclosing π^* and no one else will add more to π^*. Consequently there will never be disclosures greater than π^* in equilibrium.

Proposition 5. *For any protocol, there is always an undominated disclosure that is not greater than* π^**.*

Proposition 5 follows from backward induction and the observation that if π^* has already been disclosed, it is not valuable to disclose more. Hence if an expert discloses π^* when it is her turn, then she knows that there will exist undominated strategies for future experts that involve no further disclosure.

Proposition 6. *If there are no ties, there is a unique disclosure that survives IDWDS.*

Corollary 3. *If there are no ties, no disclosure that survives IDWDS is greater than* π^**.*

Corollary 3 is a consequence of Proposition 5, which shows existence of a disclosure that satisfies IDWDS no greater than π^* and Proposition 6, which shows that there is no other disclosure that survives IDWDS.

When X is completely ordered (one dimensional), our results characterize what is possible using sequential protocols and provide a sense in which the canonical protocol is optimal for the decision maker. In the generic, one-dimensional case, the results of this subsection show that every protocol generates only disclosures less than or equal to π^*. Combined with Proposition 2, this means that the canonical protocol generates the largest disclosure. When X has more than one dimension, there is a difference between "less than or equal to" and "not larger than." The propositions leave open the possibility that there may be a protocol that generates a disclosure that is not comparable to π^*. The next example demonstrates that this possibility really arises.

Example 8. *There are 6 disclosures, of the form* (i, j) *for* $i = 1, 2, 3$ *and* $j = 1, 2$ *and 3 experts whose preferences are*

Expert 1: $(1, 1) \succ (1, 2) \succ (3, 1) \succ (3, 2) \succ (2, 1) \succ (2, 2)$

Expert 2: $(2, 1) \succ (1, 1) \succ (2, 2) \succ (3, 1) \succ (1, 2) \succ (3, 2)$

Expert 3: $(1, 1) \succ (1, 2) \succ (2, 1) \succ (3, 1) \succ (3, 2) \succ (2, 2)$

These preferences satisfy quasi-supermodularity. In this case, $\tilde{\pi}^* = \pi^* = \bar{\pi}^* = (3, 1)$*. Expert 1's favorite disclosure is* $(1, 1)$*, but if she discloses* $(1, 1)$*, then Expert 2 will disclose* $(2, 1)$*, which will be the final outcome. If Expert 1 discloses* $(1, 2)$*, then this will be the final outcome (if Expert 2 discloses more, full disclosure of* $(3, 2)$ *will be the outcome). Hence the protocol generates disclosure* $(1, 2)$*.*

This result is consistent with the results of Section 5.4. The protocol generates a unique disclosure and this disclosure is not greater than π^**. It demonstrates that it may be possible to generate a disclosure that is not comparable to* π^**. Hence there will be some decision maker who prefers more disclosure to less but also prefers the protocol of asking Expert 1, then 2, then 3 to the canonical sequential protocol (that leads to the disclosure* π^**).*

5.5 Comparative Statics

In this section we make a few observations about the value of adding experts.

Adding an expert cannot harm the decision maker in the sense that if π is a disclosure for the original players that survives IDWDS in the simultaneous game, a disclosure of at least π will survive if additional players are added and a new player need not be consulted in a sequential protocol. Li and Norman (2018b) show that adding an expert may hurt the decision maker if the expert must be inserted in a particular place.

In situations in which our bounds on disclosure are tight ($\bar{\pi}^* = \tilde{\pi}^* = \pi^*$) adding an additional expert can only be beneficial if doing so increases one of these quantities. An expert that does not increase one of these quantities is **redundant**. It is clear that if preferences are single-peaked, all experts except the one with the greatest peak is redundant. More generally, if there are a pair of experts i and j such that $\bar{v}_i(x) \geq \bar{v}_j(x)$ for all x, then Expert j is redundant.

6 Comparing Disclosures in Simultaneous and Sequential Games

We have described the set of disclosures that can survive IDWDS in the simultaneous games and can be outcomes of a sequential disclosure protocol. When $\tilde{\pi}^* = \pi^* = \bar{\pi}^*$, we can assert that the decision maker does equally well with either form of consultation. This means that for generic games with one-dimensional strategy spaces there is no obvious reason to favor sequential versus simultaneous disclosure. In general, the different forms of consultation may perform differently. Sometimes simultaneous disclosure games enable larger disclosures than sequential games, sometimes sequential disclosure will be superior from the perspective of the decision maker.

When $\pi^* < \bar{\pi}^*$, the simultaneous game may admit equilibria surviving IDWDS that disclose strictly more than π^* (Example 6). In this example the only possible sequential disclosure is π^* because all disclosures are comparable to π^*. Hence, it is possible for simultaneous disclosure to strictly exceed the upper bound for disclosure in any sequential protocol.

When $\tilde{\pi}^* < \pi^* \leq \bar{\pi}^*$, the following results demonstrate that if there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, then π^* will survive in some equilibrium outcome that survives IDWDS.

Proposition 7. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a procedure of IDWDS in the game determined by a sequential disclosure protocol so that the disclosure π^* survives.*

Proposition 7 does not imply that there is a sequential disclosure protocol that guarantees only π^* survives IDWDS for all orders of deleting strategies. This stronger statement is not true. In the proof of the proposition it is important that we delete dominated strategies of the initial player (Expert i^*) only after other weakly dominated strategies are deleted. Otherwise, it is possible that initial disclosures from Expert i^* greater than $\tilde{\pi}^*$ will be deleted.

If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, Example 1 demonstrates that π^* need not survive all procedures of IDWDS in the simultaneous game. One can check that in this game there is no sequential protocol that guarantees that π^* is the unique disclosure that survives IDWDS.

Proposition 8. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a procedure of IDWDS in the simultaneous game so that the disclosure π^* survives.*

By Proposition 7, there is a sequential disclosure protocol that generates π^* if an expert is indifferent between $\tilde{\pi}^*$ and π^*. By Proposition 8, π^* also survives IDWDS in the simultaneous game under this condition. If all experts, however, strictly prefer $\tilde{\pi}^*$ to π^*, π^* may not survive IDWDS in either the simultaneous or the sequential game.

Even when there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, sequential and simultaneous consultation still need not be equivalent, because π^* may be the unique disclosure that survives IDWDS in the simultaneous game, but not the sequential game or in the sequential game, but not simultaneous game.

Finally, if there is no expert who is indifferent between $\tilde{\pi}^*$ and π^*, the next example illustrates that it is possible to construct a sequential protocol that induces the disclosure π^* in an equilibrium, but that the disclosures surviving IDWDS in the simultaneous game are strictly smaller. This means that there are situations in which a sequential protocol may be more attractive than simultaneous disclosure.

Example 9. *Consider the following game:*

	None	*Some*	*More*	*All*
None	3, 2	0, 2	2, 1	1, 1
Some	0, 2	0, 2	2, 1	1, 1
More	2, 1	2, 1	2, 1	1, 1
All	1, 1	1, 1	1, 1	1, 1

It is straightforward to confirm that $\tilde{\pi}^* = None$ and $\pi^* = All$ and that any order of deletion of weakly dominated strategies leaves $\{None, More\}$ for Expert 1 and $\{None, Some\}$ for Expert 2. Consequently, the disclosure in the simultaneous game is always less than All. In contrast, when Expert 1 discloses first and Expert 2 follows, there is an outcome in undominated strategies in which Expert 1 discloses More and Expert 2 discloses All when Expert 1 discloses More, Some when Expert 1 discloses None, and otherwise does not disclose more than Expert 1. Expert 2's strategy is not weakly dominated because it plays a best response at each information set. Expert 1's only best replies to Expert 2's strategies involve disclosing More or All, so More cannot be weakly dominated.[8] *Hence there is a sequential protocol that generates a greater disclosure than what is possible in the simultaneous game.*

[8] Expert 1's strategy *All* cannot dominate *More*, because *More* is the unique best reply to another (undominated) strategy of Expert 2. This strategy discloses *Some* when Expert 1 discloses *None* and otherwise not disclosing more than what Expert 1 has already disclosed.

7 Discussion

One of our goals in this study was to identify features that favored simultaneous versus sequential disclosure. Our results suggest that the choice of organization does not matter, at least in a leading case. Nevertheless, even in our setting, there are differences between the procedures. The sequential protocol that we identify requires that the decision maker know a lot about the preferences of the experts. The fine details of the optimal protocol depend on these preferences. In contrast, the simultaneous protocol uses the same game to induce disclosure independent of the preferences of the experts. This observation makes the simultaneous protocol more appealing. Both organizations require that the experts know a lot about the preferences of other experts. It is not clear whether uncertainty about preferences would have a differential impact on performance. In one way, sequential disclosure may be superior to simultaneous disclosure. Simultaneous disclosure requires that all experts (or, at least, an essential subset of experts) participate. Under the optimal sequential protocol, however, there exists an order of deleting weakly dominated strategies such that the first expert discloses $\tilde{\pi}^*$.[9] Consequently, if it is costly to consult experts, a sequential protocol might be more attractive than a simultaneous one.

There appear to be natural settings in which simultaneous disclosure is preferred and other settings in which sequential procedures are the norm. On one hand, editors typically consult several reviewers simultaneously to obtain reports on a submission. Committee deliberations, anonymous voting, and obtaining multiple bids for a construction contract have features of simultaneous disclosure. On the other hand, it is frequent to consult medical experts in sequence. A full understanding of the relative merits of simultaneous and sequential procedures therefore requires modifying our model. Two directions seem promising. Certainly adding costs (whether direct payment or waiting times) to consultation will change the analysis, presumably in the direction of favoring sequential procedures. Somewhat less obvious, in our model experts cannot learn from each other. It would be interesting to study a variation of the model in which experts had access to different facts or where the disclosure of one expert influences the set of disclosures available to subsequent agents.

We have studied two ways to organize consultations. It is natural to ask whether there is an organization that is superior to either the simultaneous or sequential procedures that we examined. We lack a complete answer to this question, which differs from standard implementation questions in two ways. First, we must study something more general than implementation in Nash equilibrium (because otherwise it is straightforward to obtain full disclosure). Second, in the standard implementation problem, the decision maker's set of actions is fixed. We assume that what the decision maker can do depends on the actual disclosures of the experts (the decision maker cannot take an action unless an expert identifies it). Although we lack a general formulation, given that all agents disclose no more than π^*, there is never an incentive for a particular agent to disclose more than π^*, at least in the one-dimensional case. This observation suggests that any procedure that eliminates weakly dominated strategies cannot induce a disclosure greater than π^*.

[9]The first expert has a strategy that discloses $\tilde{\pi}^*$ that survives IDWDS independent of the order of deletion.

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Appendix

Lemma 1. *If π' and π'' are equilibrium disclosures, then $\pi' \wedge \pi''$ is an equilibrium disclosure.*

Proof. If π' and π'' are equilibrium disclosures, then $u_i(\pi'') \geq u_i((\pi' \wedge \hat{\pi}) \vee \pi'')$ and $u_i(\pi') \geq u_i(\pi' \vee \hat{\pi})$ for any $\hat{\pi}$. It follows from quasi-supermodularity that

$$u_i((\pi' \wedge \hat{\pi}) \wedge \pi'') \geq u_i(\pi' \wedge \hat{\pi}) \tag{3}$$

and

$$u_i(\pi' \wedge \hat{\pi}) \geq u_i(\hat{\pi}). \tag{4}$$

Now assume that $\hat{\pi} > \pi' \wedge \pi''$. Consequently, $(\pi' \wedge \hat{\pi}) \wedge \pi'' = \pi' \wedge \pi''$ so that inequality (3) implies

$$u_i(\pi' \wedge \pi'') \geq u_i(\pi' \wedge \hat{\pi}). \tag{5}$$

It follows from (4) and (5) that $u_i(\pi' \wedge \pi'') \geq u_i(\hat{\pi})$ so that $\pi' \wedge \pi''$ is an equilibrium disclosure. □

Claim 1. *Let π be a disclosure that satisfies $u_i(\pi(k), \bar{x}(-k)) > u_i(x_i(k), \bar{x}(-k))$ for all $x_i(k) > \pi(k)$, all dimensions k, and all i. Then π is an equilibrium disclosure such that $u_i(\pi) > u_i(x_i)$ for all $x_i > \pi$ and all i.*

Proof. Fix a disclosure $\pi' > \pi$ and an arbitrary expert i. Then $\pi'(k) \geq \pi(k)$ on all dimensions and the inequality is strict on at least one dimension. We will show that $u_i(\pi) > u_i(\pi')$. Without loss of generality, suppose that $\pi'(1) > \pi(1)$, where $\pi(1)$ is the disclosure of π on dimension 1 and $\pi(-1)$ is the disclosure of π on dimensions other than 1. We will show by induction on r that

$$u_i(\pi) > u_i(\pi'(1), \ldots, \pi'(r), \pi(r+1), \ldots, \pi(n)) \tag{6}$$

for $r = 1, \ldots, n$. The claim follows from inequality (6) for $r = n$.

Because $u_i(\pi(1), \bar{x}(-1)) > u_i(\pi'(1), \bar{x}(-1))$, it must be that $u_i(\pi) > u_i(\pi'(1), \pi(-1))$ by quasi-supermodularity. Hence inequality (6) holds for $r = 1$. Assume inequality (6) for $r \leq k$. It suffices to show that it holds for $r = k+1$.

Because $u_i(\pi(k+1), \bar{x}(-(k+1))) \geq u_i(\pi'(k+1), \bar{x}(-(k+1)))$,

$$u_i(\bar{x}(1), \ldots, \bar{x}(k), \pi(k+1), \ldots, \pi(n)) \geq u_i(\bar{x}(1), \ldots, \bar{x}(k), \pi'(k+1), \pi(k+2), \ldots, \pi(n)) \tag{7}$$

by quasi-supermodularity. Inequality (7) and quasi-supermodularity imply that

$$u_i(\pi'(1), \ldots, \pi'(k), \pi(k+1), \ldots, \pi(n)) \geq u_i(\pi'(1), \ldots, \pi'(k+1), \pi(k+2), \ldots, \pi(n)). \tag{8}$$

It follows from inequality (6) for $r = k$ and inequality (8) that inequality (6) holds for $r = k+1$, which completes the proof. □

Proposition 1. *If x is a strategy profile that survives IDWDS, then $M(x) \in [\tilde{\pi}^*, \bar{\pi}^*]$. If x is a Nash equilibrium strategy profile that survives IDWDS, then $\tilde{u}_i(x) \geq u_i(\bar{\pi}^*)$ for all i.*

Denote the set of strategies that survive IDWDS by S. We prove the proposition in a series of steps, which we state and prove as claims.

Claim 2. *For all $x \in S$ and every i, there exists $x_i \in S_i$ that is a best response to x relative to X.*

Proof. The result is clear if the best response to x has not yet been deleted. If the best response to x has been deleted, then it was deleted by a strategy that weakly dominates it. This strategy must be a best reply to x. $\square$

Claim 3. *There exists a strategy profile $x \in S$ such that $M(x_1, \ldots, x_I) \leq \pi^*$.*

Proof. Suppose that after r iterations, there exists a strategy profile x satisfying the condition in the claim. In the next iteration, every agent must have a strategy that is a best response to x by Claim 2. Suppose that the best response for Expert i to x is some $y_i \not\leq \pi^*$. We will argue to a contraction. Because $M(y_i, x_{-i}) = y_i \vee M(x_{-i}) = M(y_i \vee M(x_{-i}), x_{-i})$,

$$\tilde{u}_i(y_i, x_{-i}) = u_i(M(y_i, x_{-i})) = u_i(M(y_i \vee M(x_{-i}), x_{-i})) = \tilde{u}_i(y_i \vee M(x_{-i}), x_{-i}). \tag{9}$$

It follows from (9) that $y_i' \equiv y_i \vee M(x_{-i})$ is also a best response to x. Note that because $y_i' > M(x_{-i})$,

$$M(y_i', x_{-i}) = y_i'. \tag{10}$$

In addition, note that

$$M(y_i' \wedge \pi^*, x_{-i}) = (y_i' \wedge \pi^*) \vee M(x_{-i}) = y_i' \wedge \pi^*, \tag{11}$$

where second equality holds because $M(x_{-i}) < y_i'$ and $M(x_{-i}) \leq \pi^*$ imply that $M(x_{-i}) \leq y_i' \wedge \pi^*$.

By definition of π^*, $u_i(\pi^*) > u_i(x_i)$ for all $x_i > \pi^*$. Because $y_i \not\leq \pi^*$ and $y_i' \geq y_i$, $y_i' \not\leq \pi^*$ and therefore $y_i' \vee \pi^* > \pi^*$. It follows that

$$u_i(\pi^*) > u_i(y_i' \vee \pi^*). \tag{12}$$

Hence it must be that

$$\tilde{u}_i(y_i' \wedge \pi^*, x_{-i}) = u_i(M(y_i' \wedge \pi^*, x_{-i})) = u_i(y_i' \wedge \pi^*) > u_i(y_i') = u_i(M(y_i', x_{-i})) = \tilde{u}_i(y_i', x_{-i}) \tag{13}$$

where the second equation follows from (11), the inequality follows by (12) and quasi-supermodularity, and the third equation follows from (10). Expressions (9) and (13) combine to show that y_i is not a best response to x_{-i}, which is a contradiction. We conclude that no strategy $y_i' \not\leq \pi^*$ can do at least as well as $y_i' \wedge \pi^*$ against x. Therefore, all best responses to x are less than or equal to π^* and a strategy less than or equal to π^* must remain. $\square$

Then we show that all disclosures that survive IDWDS are greater than or equal to the smallest equilibrium disclosure.

Claim 4. *If $x \in S$, then $M(x) \geq \tilde{\pi}^*$.*

Proof. Let $\tilde{\pi} \equiv \wedge_{x \in S} M(x)$ be the meet of all disclosures that survive IDWDS. We wish to show that $\tilde{\pi} \geq \tilde{\pi}^*$. In order to reach a contradiction, assume that $\tilde{\pi} \not\geq \tilde{\pi}^*$. By the definition of $\tilde{\pi}^*$, it must be the case that for some i,

$$\text{there exists } x_i \text{ with } x_i > \tilde{\pi} \text{ such that } u_i(x_i) > u_i(\tilde{\pi}), \tag{14}$$

because otherwise $\tilde{\pi}^*$ would not be the smallest equilibrium disclosure. Let

$$y_i = \min\{\arg\max_{x_i \geq \tilde{\pi}} u_i(x_i)\}$$

be the smallest best response of Player i to disclosure $\tilde{\pi}$, which is well defined by Fact 1. It is apparent that $y_i > \tilde{\pi}$ and so $y_i(k) > \tilde{\pi}(k)$ for some k. Select $\tilde{x} \in S$ such that $M(\tilde{x})(k) = \tilde{\pi}(k)$. This is possible by the definition of $\tilde{\pi}$. We assert that $y_i \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. The assertion is sufficient to prove the claim because it means that if (14) holds, then we can show an element in S is weakly dominated. This contradicts the definition of S. It remains to show that $y_i \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. For any x_{-i} such that $M(x_{-i}) \geq y_i \vee \tilde{x}_i$,

$$\tilde{u}_i(y_i \vee \tilde{x}_i, x_{-i}) = u_i(M(x_{-i})) = \tilde{u}_i(\tilde{x}_i, x_{-i}).$$

For any x_{-i} such that $M(x_{-i}) \not\geq y_i \vee \tilde{x}_i$, i's utility from playing $y_i \vee \tilde{x}_i$ and $\tilde{x}_i$ are $u_i((y_i \vee \tilde{x}_i) \vee M(x_{-i})) = u_i(y_i \vee (\tilde{x}_i \vee M(x_{-i})))$ and $u_i(\tilde{x}_i \vee M(x_{-i}))$, respectively. We now claim that

$$\tilde{u}_i(y_i \vee \tilde{x}_i, x_{-i}) = u_i(y_i \vee (\tilde{x}_i \vee M(x_{-i}))) \geq u_i(\tilde{x}_i \vee M(x_{-i})) = \tilde{u}_i(\tilde{x}_i, x_{-i}). \tag{15}$$

Hence $y_i \vee \tilde{x}_i$ is weakly better than $\tilde{x}_i$. The equations in (15) follow from the definition of $\tilde{u}_i$. The inequality holds by quasi-supermodularity because

$$u_i(y_i) \geq u_i(y_i \wedge (\tilde{x}_i \vee M(x_{-i}))). \tag{16}$$

Note that $\tilde{x}_i \vee M(x_{-i}) \geq \tilde{\pi}$ by the definition of $\tilde{\pi}$ and $y_i > \tilde{\pi}$, imply that $y_i \wedge (\tilde{x}_i \vee M(x_{-i})) \geq \tilde{\pi}$. Inequality (16) now follows from the definition of y_i.

It follows from (14) and the definition of y_i that $y_i > \tilde{\pi}$. Also $y_i > y_i \wedge M(\tilde{x}) \geq \tilde{\pi}$, and so the definition of y_i also implies that $u_i(y_i) > u_i(y_i \wedge M(\tilde{x}))$. Consequently, $u_i(y_i \vee M(\tilde{x})) > u_i(M(\tilde{x}))$ by quasi-supermodularity. It follows that

$$\tilde{u}_i(y_i \vee \tilde{x}_i, \tilde{x}_{-i}) = u_i(y_i \vee M(\tilde{x})) > u_i(M(\tilde{x})) = \tilde{u}_i(\tilde{x}) = \tilde{u}_i(\tilde{x}_i, \tilde{x}_{-i}). \tag{17}$$

Inequality (17) guarantees that $y_i \vee \tilde{x}_i$ is strictly better than $\tilde{x}_i$ when $x_{-i} = \tilde{x}_{-i}$. Establishing this completes the proof that $y_i \vee \tilde{x}_i$ weakly dominates $\tilde{x}_i$. □

All disclosures that survive IDWDS are greater than or equal to $\tilde{\pi}^*$. Next, we show that all disclosures that survive IDWDS are less than or equal to the bounding disclosure.

Claim 5. *If $x \in S$, then $M(x) \leq \bar{\pi}^*$.*

Proof. Let S_i^r be the set of strategies remaining for i after r rounds of deleting strategies. For each i let $P_i^r = \{s_i^r \in S_i^r : s_i^r \not\leq \bar{\pi}^*\}$. If there exists r such that $\bigcup_i P_i^r = \emptyset$, then the proof is complete. Otherwise, there is at least one dimension k such that $s_i^r(k) > \bar{\pi}^*(k)$ for some $s_i^r \in P_i^r$ and some i. Let $Q^r(k) = \bigcup_i \{s_i^r \in P_i^r : s_i^r(k) > \bar{\pi}^*(k)\}$. Let $z^r \in \arg\min\{s^r(k) : s^r \in Q^r(k)\}$. If $\bigcup_i P_i^r \neq \emptyset$, then z^r exists and $z^r \in P_j^r$ for some j; we write $z^r = z_j^r$ to indicate that $z_j^r \in P_j^r$. We claim that z_j^r is weakly dominated by $(\bar{\pi}^*(k), z_j^r(-k))$.[10]

For any x such that $M(x_{-j}(k)) \geq z_j^r(k)$,

$$\tilde{u}_j((\bar{\pi}^*(k), z_j^r(-k)), x_{-j}) = u_j(M(x_{-j}(k)), M(z_j^r(-k), x_{-j}(-k))) = \tilde{u}_j(z_j^r, x_{-j}).$$

This follows because $z_j^r(k) > \bar{\pi}^*(k)$.

For any x such that $M(x_{-j}(k)) < z_j^r(k)$, $M(x_{-j}(k)) \leq \bar{\pi}^*(k)$ by the definition of z_j^r. Hence, Expert j's utility from using z_j^r is $u_j(z_j^r(k), M(z_j^r(-k), x_{-j}(-k)))$, while j's utility from using $(\bar{\pi}^*(k), z_j^r(-k))$ is $u_j(\bar{\pi}^*(k), M(z_j^r(-k), x_{-j}(-k)))$. We claim that

$$u_j(\bar{\pi}^*(k), M(z_j^r(-k), x_{-j}(-k))) > u_j(z_j^r(k), M(z_j^r(-k), x_{-j}(-k))). \tag{18}$$

It follows from the definition of $\bar{\pi}^*$ that $u_j(\bar{\pi}^*(k), \bar{x}(-k)) > u_j(z_j^r(k), \bar{x}(-k))$, therefore Inequality (18) follows from quasi-supermodularity. That is, j does strictly better using $(\bar{\pi}^*(k), z_j^l(-k))$ than z_j^r whenever the k^{th} dimension of $M(x_{-j})$ is less than $z_j^r(k)$. Because there always exists a strategy in which $M(x_{-j}(k))$ is less than $z_j^r(k)$ by Claim 3 and $\pi^* \leq \bar{\pi}^*$, $(\bar{\pi}^*(k), z_j^r(-k))$ must be strictly better than z_j^r against one strategy profile that survives IDWDS. Consequently, $(\bar{\pi}^*(k), z_j^r(-k))$ weakly dominates z_j^r. Therefore, z_j^r must eventually be deleted. We conclude that there must exist a r^* such that $P_i^{r^*} = \emptyset$ for all i, which establishes the result. □

Proof of Proposition 1. Claim 4 guarantees that all $x \in S$ satisfy $M(x) \geq \tilde{\pi}^*$. Claim 5 guarantees that all $x \in S$ satisfy $M(x) \leq \bar{\pi}^*$. This establishes the first part of the Proposition. Given any $x \in S$, it follows from Claim 2 that each player has a surviving strategy that is a best response to x_{-i} relative to the full strategy set. Because $x_i = \bar{\pi}^*$ leads to payoff $u_i(\bar{\pi}^*)$ for Expert i against any surviving strategy by Claim 5, the second part of the proposition follows. □

Lemma 2. *$\pi^m < \tilde{\pi}^*$ for $m < \bar{m}$ and $\pi^{\bar{m}} = \tilde{\pi}^*$.*

Proof. It is sufficient to show that $\pi^m \leq \tilde{\pi}^*$ for all m. Suppose that $\pi^m \not\leq \tilde{\pi}^*$ for some m. Let $\pi^{m'}$ be the disclosure such that $\pi^{m'} \not\leq \tilde{\pi}^*$ and $\pi^{m'-1} \leq \tilde{\pi}^*$. Because $\pi^0 = \underline{x} \leq \tilde{\pi}^*$, m' is well defined. The fact that $\pi^{m'} \not\leq \tilde{\pi}^*$, implies that $\pi^{m'} \wedge \tilde{\pi}^* < \pi^{m'}$. By quasi-supermodularity of $\underline{i}(m')$'s utility function, $u_{\underline{i}(m')}(\tilde{\pi}^*) \geq u_{\underline{i}(m')}(\pi^{m'} \vee \tilde{\pi}^*)$ implies that $u_{\underline{i}(m')}(\pi^{m'} \wedge \tilde{\pi}^*) \geq u_{\underline{i}(m')}(\pi^{m'})$. Further, note that $\pi^{m'-1} \leq \tilde{\pi}^*$ implies that $\pi^{m'-1} \leq \pi^{m'} \wedge \tilde{\pi}^*$. Hence $\pi^{m'}$ cannot be the smallest disclosure of Expert $\underline{i}(m')$ that maximizes her payoff given that $\pi^{m'-1}$ has already been disclosed. This contradicts the definition of $\pi^{m'}$, which means that $\pi^m < \tilde{\pi}^*$ for $m < \bar{m}$ and $\pi^{\bar{m}} = \tilde{\pi}^*$. □

[10] If $X \subset \mathbb{R}$, $(\bar{\pi}^*(k), z_j^r(-k)) = \bar{\pi}^* = \pi^*$.

Proposition 2. *If π is a disclosure that survives IDWDS in the game determined by the canonical disclosure protocol, then $\pi \geq \tilde{\pi}^*$.*

Proof. We will show by induction on m that if $x \in S$, then $\mu(h_{\bar{m}}(x)) \geq \pi^m$ for $m = 0, \ldots, \bar{m}$. The proposition follows because $\pi^{\bar{m}} = \tilde{\pi}^*$. Suppose that $\mu(h_{\bar{m}}(x)) \geq \pi^{m'-1}$ for all $x \in S$ and $1 \leq m' \leq \bar{m}$. It suffices to show that $\mu(h_{\bar{m}}(x)) \geq \pi^{m'}$ for all $x \in S$.

Consider a disclosure $\hat{\pi} \geq \pi^{m'-1}$ such that $\hat{\pi} \not\geq \pi^{m'}$ and suppose that there exists a strategy profile $\hat{x}$ such that $\mu(h_{\bar{m}}(\hat{x})) = \hat{\pi}$. We will show that there exists an expert i such that $\hat{x}_i$ is weakly dominated.

For all $\pi \not\geq \pi^{m'}$, $\pi^{m'} \geq \pi^{m'-1}$ and $\pi \geq \pi^{m'-1}$ imply

$$\pi^{m'} > \pi^{m'} \wedge \pi \geq \pi^{m'-1}. \tag{19}$$

The definition of $\pi^{m'}$ implies $u_{\underline{i}(m')}(\pi^{m'} \wedge \pi) < u_{\underline{i}(m')}(\pi^{m'})$ and therefore, by quasi-supermodularity,

$$u_{\underline{i}(m')}(\pi^{m'} \vee \pi) > u_{\underline{i}(m')}(\pi) \text{ for all } \pi \not\geq \pi^{m'}, \pi \geq \pi^{m'-1}. \tag{20}$$

Let $\check{x}$ solve: $\max u_{\underline{i}(m')}(\pi^{m'} \vee \mu(h_{\bar{m}}(x)))$ subject to $x \in S, h_{\bar{m}-m'}(x) = h_{\bar{m}-m'}(\hat{x})$. For all $x \in S$ such that $h_{\bar{m}-m'}(x) = h_{\bar{m}-m'}(\hat{x})$, we have

$$u_{\underline{i}(m')}(\pi^{m'} \vee \mu(h_{\bar{m}}(\check{x}))) \geq u_{\underline{i}(m')}(\pi^{m'} \vee \mu(h_{\bar{m}}(x))) \geq u_{\underline{i}(m')}(\mu(h_{\bar{m}}(x))). \tag{21}$$

The second inequality in (21) is an equation when $\mu(h_{\bar{m}}(x)) \geq \pi^{m'}$. Otherwise, it follows from (20) because $\mu(h_{\bar{m}}(x)) \geq \pi^{m'-1}$ by the induction hypothesis.

Consider an alternative strategy of Expert $\underline{i}(m')$, for $P(h) = \underline{i}(m')$,

$$x'_{\underline{i}(m')}(h) = \begin{cases} \pi^{m'} \vee \mu(h_{\bar{m}}(\check{x})) & \text{if } h = h_{\bar{m}-m'}(\hat{x}) \\ \hat{x}_{\underline{i}(m')}(h) & \text{if } h \neq h_{\bar{m}-m'}(\hat{x}) \end{cases}.$$

It follows that $x'_{\underline{i}(m')}$ weakly dominates $\hat{x}_{\underline{i}(m')}$. The strategy $x'_{\underline{i}(m')}$ does exactly as well as $\hat{x}_{\underline{i}(m')}$ for $h \neq h_{\bar{m}-m'}(\hat{x})$. When $h = h_{\bar{m}-m'}(\hat{x})$, the decision maker asks Expert $\underline{i}(m')$ to make a disclosure at stage $\bar{m} - m' + 1$ because $\hat{\pi} \not\geq \pi^{m'}$. Hence, by Inequality (21), $x'_{\underline{i}(m')}$ is weakly better than $\hat{x}_{\underline{i}(m')}$. That $x'_{\underline{i}(m')}$ is strictly better when $\mu(h_{\bar{m}}(x)) = \hat{\pi}$ follows from (20) and (21), because $\pi^{m'} \vee \mu(h_{\bar{m}}(\check{x})) \geq \pi^{m'}$ guarantees that for $h = h_{\bar{m}-m'}(\hat{x})$ the decision maker stops asking experts after Expert $\underline{i}(m')$ discloses $\pi^{m'}$ in stage $\bar{m} - m' + 1$.

We conclude that for all $m' \leq \bar{m}$, a disclosure $\hat{\pi} \not\geq \pi^{m'}$ can only be generated if a player uses a weakly dominated strategy. This establishes the proposition. □

Lemma 3. *If $\tilde{\pi}^* \leq \pi$ and $\pi \not\leq \pi^*$, then $u_i(\pi) < u_i(\tilde{\pi}^*)$.*

Proof. It follows from the assumptions that $\tilde{\pi}^* \leq \pi \wedge \pi^* < \pi$ and $\pi \vee \pi^* > \pi^*$. Consequently, $u_i(\pi^*) > u_i(\pi^* \vee \pi)$ by the definition of π^* and $u_i(\pi^* \wedge \pi) > u_i(\pi)$ by quasi-supermodularity. The lemma follows because $u_i(\tilde{\pi}^*) \geq u_i(\pi^* \wedge \pi)$ by the definition of $\tilde{\pi}^*$. □

Proposition 3. *If π is generated by the canonical protocol, then $\pi \leq \pi^*$.*

Proof. By Lemma 3, Expert $\underline{i}(\bar{m})$ can do strictly better than any disclosure π such that $\tilde{\pi}^* \leq \pi$ and $\pi \nleq \pi^*$ by disclosing $\tilde{\pi}^*$. This disclosure ends the protocol. By Proposition 2, all disclosures generated by the canonical sequence are greater than or equal to $\tilde{\pi}^*$. This establishes the proposition. □

Proposition 5. *For any protocol, there is always an undominated disclosure that is not greater than* π^*.

Proof. Fix a protocol. Because anything greater than π^* is strictly worse than π^* for the last expert, disclosing π^* uniquely gives her the highest possible payoff when π^* has been disclosed. Then at least one of her strategies that survive IDWDS must involve disclosing π^* when π^* has been disclosed. By backward induction, there must exist a strategy profile that survives IDWDS such that experts (except the first one) will choose π^* if π^* is already disclosed, because each expert's unique best response to history with disclosure π^* is to choose π^* when the expert expects future experts to make no additional disclosure.

So one of the final disclosures will be π^* if the first expert discloses π^*. If the strategy π^* is not weakly dominated for the first expert, there is an undominated disclosure π^*. If the first expert's strategy π^* has been deleted, it is weakly dominated by a strategy that leads to a weakly higher payoff for the first expert against all remaining strategies of the other experts. Consequently at least one disclosure that survives IDWDS does at least as well as the disclosure π^* for the first expert. By definition of π^*, the first expert strictly prefers π^* to anything greater than π^*. Hence the disclosures that survive IDWDS cannot be all greater than π^*. □

Proposition 6. *If there are no ties, there is a unique disclosure that survives IDWDS.*

Proof. First, note that if an expert is indifferent between two strategies given any strategy profile of other players, the two strategies must lead to the same disclosure because of no ties. So the other experts are indifferent between the two profiles (differing only in that expert's action) as well. Hence the **transference of decisionmaker indifference** (TDI) condition in Marx and Swinkels (1997) holds.

Furthermore, it is clear that there is a unique outcome from backward induction in this game (with no ties). Marx and Swinkels (1997) show that if an extensive form game with perfect information that satisfies TDI has a unique backward induction payoff, then any full reduction by weak dominance also contains only that payoff. Therefore there is a unique disclosure that survives IDWDS, which is the unique backward induction solution. □

Proposition 7. *If there is an expert who is indifferent between* $\tilde{\pi}^*$ *and* π^**, there is a procedure of IDWDS in the game determined by a sequential disclosure protocol so that the disclosure* π^* *survives.*

Proof. Proposition 2 guarantees the existence of a sequential disclosure protocol that generates at least the disclosure $\tilde{\pi}^*$. Find the expert i^* who is indifferent between $\tilde{\pi}^*$ and π^*. The decision maker asks this expert to report first. If the expert discloses $x \geq \tilde{\pi}^*$, then the decision maker stops. Otherwise, the decision maker follows the canonical sequential protocol. We will describe a procedure of IDWDS so that π^* survives.

We first iteratively delete all strategies of subsequent experts $\underline{i}(m)$ that are weakly dominated for $1 \leq m \leq \bar{m}$. Their strategies describe what they will play given that Expert i^* does not disclose something greater than or equal to $\tilde{\pi}^*$. By Proposition 2, all disclosures that survive IDWDS for experts $\underline{i}(1), \ldots, \underline{i}(\bar{m})$ are greater than or equal to $\tilde{\pi}^*$. So we eliminate all strategy profiles x such that $\mu(h_{\bar{m}+1}(x)) \ngeq \tilde{\pi}^*$ in the augmented game. Because the final disclosure must be at least $\tilde{\pi}^*$, there is no strategy of Expert i^* that could weakly dominate π^*, regardless of the order of deletion thereafter. So Expert i^*'s strategy π^* survives IDWDS and will lead to the disclosure π^*. □

Proposition 8. *If there is an expert who is indifferent between $\tilde{\pi}^*$ and π^*, there is a procedure of IDWDS in the simultaneous game so that the disclosure π^* survives.*

To prove the proposition we need a preliminary result. We describe a process that deletes weakly dominated strategies. The existence of this process establishes the proposition. To do this, we describe a specific procedure of IDWDS that uses some of the structure we created to study the sequential problem.

Let S_i^m be the set of remaining strategies of i after m rounds and $S^m = \prod_{i \in I} S_i^m$ be the set of strategy profiles. In the m^{th} stage ($1 \leq m \leq \bar{m}$), we delete all strategies $x_{\underline{i}(m)}$ of Expert $\underline{i}(m)$ such that $x_{\underline{i}(m)} \vee \pi^{m-1} \ngeq \pi^m$.

The next result shows that strategies of this kind are weakly dominated and shows that we can eliminate all strategy profiles x such that $M(x) \ngeq \tilde{\pi}^*$.

In the next claim, $\pi^{-1} = \underline{x}$.

Claim 6. *For all r,*

$$\text{if } x_i^r \in S_i^r \text{ for all } i, \text{ then } M(x_1^r, \ldots, x_I^r) \geq \pi^r \tag{22}$$

and

$$x_{\underline{i}(r)}^r \vee \pi^{r-1} \geq \pi^r. \tag{23}$$

Proof. The claim holds for $r = 0$. We present a proof by induction. Suppose that for $r = m - 1$, Conditions (22) and (23) hold. We will show that they hold for $r = m$. Let $x \in S^{r-1}$. We want to show that if $x_{\underline{i}(m)} \vee \pi^{m-1} \ngeq \pi^m$, then $x_{\underline{i}(m)}$ is weakly dominated by $\pi^m \vee x_{\underline{i}(m)}$. Expert $\underline{i}(m)$'s utility from playing $\pi^m \vee x_{\underline{i}(m)}$ and $x_{\underline{i}(m)}$ are

$$u_{\underline{i}(m)}((\pi^m \vee x_{\underline{i}(m)}) \vee M(x_{-\underline{i}(m)})) = u_{\underline{i}(m)}(\pi^m \vee (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)})))$$

and $u_{\underline{i}(m)}(x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))$, respectively. By the definition of π^m, $u_{\underline{i}(m)}(\pi^m) \geq u_{\underline{i}(m)}(y)$ for $\pi^m \geq y \geq \pi^{m-1}$. By the induction hypothesis for $r = m - 1$,

$$\pi^m \wedge (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)})) \geq \pi^{m-1}.$$

It follows that

$$u_{\underline{i}(m)}(\pi^m) \geq u_{\underline{i}(m)}(\pi^m \wedge (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))).$$

So $u_{\underline{i}(m)}(\pi^m \vee (x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))) \geq u_{\underline{i}(m)}(x_{\underline{i}(m)} \vee M(x_{-\underline{i}(m)}))$ by quasi-supermodularity. It follows that $\pi^m \vee x_{\underline{i}(m)}$ is weakly better than $x_{\underline{i}(m)}$. Furthermore, there is some $x_{-\underline{i}(m)}$ such that $M(x_{-\underline{i}(m)}) = \pi^{m-1}$ because $\underline{i}(m) \neq \underline{i}(m-1)$.

Because $x_{\underline{i}(m)} \vee \pi^{m-1} \ngeq \pi^m$, $u_{\underline{i}(m)}(\pi^m) > u_{\underline{i}(m)}(\pi^m \wedge (x_{\underline{i}(m)} \vee \pi^{m-1}))$ by the definition of π^m. Quasi-supermodularity implies that $u_{\underline{i}(m)}(\pi^m \vee (x_{\underline{i}(m)} \vee \pi^{m-1})) > u_{\underline{i}(m)}(x_{\underline{i}(m)} \vee \pi^{m-1})$. We conclude that $\tilde{u}_{\underline{i}(m)}(\pi^m \vee x_{\underline{i}(m)}, \pi^{m-1}) > \tilde{u}_{\underline{i}(m)}(x_{\underline{i}(m)}, \pi^{m-1})$ when $M(x_{-\underline{i}(m)}) = \pi^{m-1}$. Hence $\pi^m \vee x_{\underline{i}(m)}$ weakly dominates $x_{\underline{i}(m)}$ as claimed. We are therefore able to delete all $x_{\underline{i}(m)}$ such that $x_{\underline{i}(m)} \vee \pi^{m-1} \ngeq \pi^m$ in stage m. This establishes (23) for $r = m$. By the induction hypothesis,

$$x^m_{\underline{i}(m)} \vee M(x^{m-1}_{-\underline{i}(m)}) = x^m_{\underline{i}(m)} \vee (x^m_{\underline{i}(m)} \vee M(x^{m-1}_{-\underline{i}(m)})) \geq x^m_{\underline{i}(m)} \vee \pi^{m-1} \geq \pi^m$$

for all $x^m_{\underline{i}(m)} \in S^m_{\underline{i}(m)}$ and $x^{m-1}_{-\underline{i}(m)} \in S^{m-1}_{-\underline{i}(m)}$.

Because we only delete strategies of Expert $\underline{i}(m)$ in stage m, $S^m_i = S^{m-1}_i$ for all $i \neq \underline{i}(m)$. It follows that

$$\max\{x^m_1, \ldots, x^m_I\} = x^m_{\underline{i}(m)} \vee M(x^m_{-\underline{i}(m)}) \geq \pi^m$$

for all $x^m \in S^m$ because we have demonstrated that (23) holds for $l = m$. Hence all disclosures $M(x^m)$ for $x^m \in S^m$ are greater than or equal to π^m, which establishes (22) for $r = m$. □

Proof of Proposition 8. Let j be the expert who is indifferent between $\tilde{\pi}^*$ and π^*. We show that j's strategy π^* survives IDWDS, i.e. $\pi^* \in S_j$. Consider a different strategy $x_j \in S^{\bar{m}}_j$ of Expert j and a strategy $x_{-j} \in S^{\bar{m}}_{-j}$ of other players that remain after stage $\bar{m}$. Expert j's payoffs from playing π^* and x_j are $u_j(\pi^* \vee M(x_{-j}))$ and $u_j(x_j \vee M(x_{-j}))$, respectively.

First assume that $x_j \leq \pi^*$, then $\pi^* \vee M(x_{-j}) = (\pi^* \vee x_j) \vee M(x_{-j}) = \pi^* \vee (x_j \vee M(x_{-j}))$. Furthermore, the disclosure $x_j \vee M(x_{-j}) \geq \tilde{\pi}^*$ by Claim 6. So $\pi^* \wedge (x_j \vee M(x_{-j})) \geq \tilde{\pi}^*$. It follows that $u_j(\pi^*) = u_j(\tilde{\pi}^*) \geq u_j(\pi^* \wedge (x_j \vee M(x_{-j})))$ by the definition of $\tilde{\pi}^*$. This implies $u_j(\pi^* \vee M(x_{-j})) = u_j(\pi^* \vee (x_j \vee M(x_{-j}))) \geq u_j(x_j \vee M(x_{-j}))$ by quasi-supermodularity. So Expert j can never get a lower payoff from π^* than any $x_j \leq \pi^*$.

Now assume that $x_j \nleq \pi^*$. By Claim 3, there is a strategy profile x that survives IDWDS such that $M(x) \leq \pi^*$. Then there exists some $x'_{-j} \in S_{-j}$ such that $M(x'_{-j}) \leq \pi^*$. It follows that $\pi^* \vee M(x'_{-j}) = \pi^*$ and $x_j \vee M(x'_{-j}) \nleq \pi^*$, which implies that $\pi^* \vee (x_j \vee M(x'_{-j})) > \pi^*$. It follows that $u_j(\pi^*) > u_j(\pi^* \vee (x_j \vee M(x'_{-j})))$ by the definition of π^*. This implies $u_j(\pi^* \wedge (x_j \vee M(x'_{-j}))) > u_j(x_j \vee M(x'_{-j}))$ by quasi-supermodularity. By Claim 6, $x_j \vee M(x'_{-j}) \geq \tilde{\pi}^*$. It follows that

$$u_j(\pi^* \vee M(x'_{-j})) = u_j(\pi^*) = u_j(\tilde{\pi}^*) \geq u_j(\pi^* \wedge (x_j \vee M(x'_{-j}))) > u_j(x_j \vee M(x'_{-j}))$$

by the definition of $\tilde{\pi}^*$. So there is no $x_j \nleq \pi^*$ that dominates the strategy π^*.

Hence there is no strategy that could weakly dominate π^*, regardless of the order of deletion after stage $\bar{m}$. So $\pi^* \in S_j$. Because $M(x'_{-j}) \leq \pi^*$, the disclosure $\pi^* = \pi^* \vee M(x'_{-j})$ survives IDWDS and the result follows. □