The Comparative Statics of Sorting[*]

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(in progress)

Abstract

The Becker solved the marriage model for the extreme cases of supermodular or submodular match payoffs. The optimal matching for the general marriage model — the Monge-Kantorivich "transportation problem" — has seen much partial progress, but amazingly remains an open question since 1781.

Rather than solve it, we instead characterize when matching grows more assortative. We first show that a stochastic order on bivariate cdf's known as positive quadrant dependence (PQD) exactly captures the economics of increasing sorting: upward shifts lead to higher correlation of match partners for instance.

We rewrite total output in terms of *synergy*, namely, the local cross partial difference (or derivative). We then prove that two types of monotone productive changes result in increased sorting by the PQD measure: sorting increases if synergy is upcrossing or downcrossing in types and either (1) everywhere increases or (2) upcrosses through zero, and proportionately upcrosses too.

Our proofs exploit induction in the finite case, and apply to the continuum model by continuity. Our theory subsumes a wealth of famous examples of matching models that do not obey Becker's assumptions.

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1 Introduction

Assortative matching is the pre-eminent theme in the pairwise matching literature. Becker's 1973 finding is the most cited takeout messages of this literature that broadly subsumes as special cases marriage, employment, partnerships, the assignment model, pairwise trade, and even the double auction. Yet the conclusion is quite strong — higher "men" always match with higher "women", without exception — and arises under a very strong assumption that output be *supermodular* in types: i.e., positively sorting any two men and two women yields a higher total output than reverse sorting them.

Since perfectly assortative matching is an idealization, how should we understand deviations from it? Shimer and Smith (2000) essentially asked if these can be seen as evidence of search frictions. They found that a weaker sorting conclusion — matching sets increase in the strong set order as one's type rises — holds under stronger complementarity assumptions. But observed deviations from perfect sorting often cannot be explained through this frictional lens. And requiring supermodular interaction is also intuitively unappealing: For there are many natural and sometimes well-cited economic matching settings where the economics mandates that supermodularity fail, as we show by example in §3 (and in Chade, Eeckhout, and Smith (2017)).

This paper develops a general theory for nonsupermodular matching models. We must surmount one major difficulty: The general solution of who matches with whom — first attacked as the *transportation problem* by Monge (1781) — is still open. This void has greatly limited the analytic reach of the matching literature in economics, and confined its reach to the extreme sorting case. In fact, we tackle this problem by bipassing it altogether. Rather than characterize the optimal matching for any production function, we exploit advances in monotone comparative statics that succeed indirectly without ever solving for the optimum. Our methods are easy to apply, and our assumptions subsume the models in the top published papers.

We first introduce an economically motivated partial order on matching measures to capture the notion of increasingly assortative. The *positive quadrant dependence* (PQD) partial order ranks bivariate measures by masses in southwest and northeast quadrants. **Lemma 2** proves that PQD yields improved sorting by three measures: diminished distance between matched partners, increasing correlation of matched partners, and higher regression coefficients of women on their match partners. All told, we seek sorting comparative statics conclusions of direct relevance to empirical economists.

For some insight, consider the six possible nonrandomized matches among three men a, b, c and three women A, B, C (Figure 1). One can verify that each man matches with a weakly closer partner in PAM, than in NAM1 or NAM3, in turn each closer

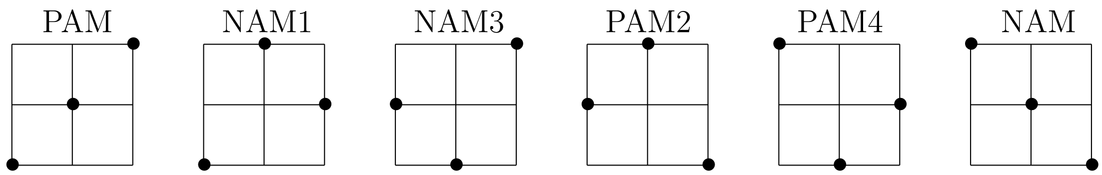

Figure 1: **All Pure Matchings with Three Types.** In addition to negative and positive assortative matching (NAM and PAM), there is negative assortative matching in quadrants 1 and 3 (NAM1 and NAM3), and positive assortative matching in quadrants 2 and 4 (PAM2 and PAM4). Sorting is partially ranked according to (1).

than in PAM2 or PAM4, and finally than in NAM. We have thus a partial order:

$$\text{PAM} \succ_{PQD} [\text{NAM1, NAM3}] \succ_{PQD} [\text{PAM2, PAM4}] \succ_{PQD} \text{NAM} \tag{1}$$

Our analysis is based on a local complementarity measure that we call *synergy* — the cross partial derivative with continuous types, and the cross partial difference with finitely many types. One way of thinking of Becker's Theorem for marriages is that if synergy is globally positive (or globally negative), then positive (or negative) sorting emerges. One might then conjecture that sorting should increase if synergy globally rises. The example in Figure 2 dashes any such hopes, since the three type matching alternates back and forth between NAM1 and NAM3 as synergy strictly rises. But neither NAM1 or NAM3 is more assortative for arbitrary weights on men and women.

To begin piecing together this story, **Lemma 1** derives a simple formula showing that total output only reflects the matching via the dot product of synergy and the cumulative match distribution. Easily, increasing synergy ensures a single crossing property. But this does not push optimizers up because the PQD order does not define a lattice. And as we've seen, when synergy increases, sorting need not. We resolve this in **Proposition 1**, exploiting a weak implication of the single crossing property alone: We argue that sorting is *nowhere decreasing* in the PQD order — it might not rise, but it never falls in the PQD order. While this result has some bite, still as synergy rises, match partners could move farther apart, or the regression coefficient on match partners could fall. This inconvenient truth highlights the need for more discipline on synergy. For instance, in the counterexample in Figure 2, synergy rises in women's types for the least man, but falls in women's types for the next man.

To deduce when sorting actually increases, we focus first on the *sorting premium* on type rectangles — namely, the net payoff change from negatively to positively sorting women $x_1 < x_2$ and men $y_1 < y_2$. This allows sharp predictions for finite type models, for which the optimal matching is generically unique: **Proposition 3** shows that generically sorting is increasing in the state when the sorting premium is strictly upcrossing in types and the state.[1] The proof is by induction on the number of types.

A parallel result holds for continuum type models, by taking limits. But our proof logic requires uniqueness. For this, we need a new property of production functions. The *x-marginal product increment* is the increase in the x-marginal product of the (x, y) match when y_1 increases to y_2. The *y-marginal product increment* is analogously defined. Exploiting recent advances in transportation theory, **Lemma 5** establishes a unique optimal matching given a monotone x- or y-marginal product increment. So equipped, **Proposition 3** shows that the very conditions for uniqueness, along with the strictly upcrossing sorting premium ensures an increasing sorting.

The sorting premium in a rectangle is a sum of synergies, and so its properties are often hard to check. We pursue a weaker general sufficient condition for it in **Lemma 6**. Synergy is *proportional upcrossing* provided any positive synergy increases proportionately more than any negative synergy absolutely increases. This result is a novel two dimensional extension of the fundamental single-crossing preservation result of Karlin and Rubin (1956). **Proposition 4** derives the monotone sorting provided that synergy either upcrosses or downcrosses through zero, cross sectionally.

Finally, given our cross-sectional assumptions, our theory affords comparative statics predictions for upward shifts in the type distributions (**Proposition 5**). Our proof exploits the formal equivalence between these type shifts and productive shifts.

This proposition greatly expands the predictive reach of matching theory. For instance, with 100 men and 100 women, Becker (1973) makes predictions for just two of the possible synergy sign combinations. Our cross-sectional single crossing synergy encompasses $2 \cdot 99^2$ sign combinations — and ones that specifically arise in applications.

Literature Review. Becker's work sparked a vast economic literature on the transferable utility matching paradigm. While the assignment aspects have escaped a general attack, some papers have pursued partial characterizations of special models without perfect sorting. Our model offers comparative statics for all of these papers.

Kremer and Maskin (1996) was an early work that made a strong case for exploring the marriage model *without supermodularity*. In this motivated twist on Becker, they proposed a partnership model with defined roles. Match output was therefore the maximum of two supermodular functions — one for each role assignment. They claim "there is a body of work within the labor economics literature that assumes such imperfect substitutability. There is also empirical evidence to justify the assumption".

Others soon highlighted the importance of matching without supermodularity.

[1] We call a real-valued function $\Upsilon(\theta)$ on a partially ordered domain, like $\mathbb{R}$ or $\mathbb{R}^n$, *upcrossing* if its sign changes at most once negative to positive: $\Upsilon(\theta) \geq 0 \Rightarrow \Upsilon(\theta') \geq 0$ and $\Upsilon(\theta) > 0 \Rightarrow \Upsilon(\theta') > 0$ for all $\theta' > \theta$. It is *downcrossing* if $-\Upsilon$ is upcrossing, and *one-crossing* if it is upcrossing or downcrossing.

Match Payoffs

	A	B	C
c	9	**14**	18
b	5	2	**14**
a	**1**	5	9

→

	A	B	C
c	9	16	**24**
b	**5**	3	16
a	1	**5**	9

→

	A	B	C
c	9	**20**	30
b	5	6	**20**
a	**1**	5	9

→

	A	B	C
c	9	22	**36**
b	**5**	7	22
a	1	**5**	9

Cross Partial Differences of Match Payoffs

	AB	BC
bc	8	−8
ab	−7	8

→

	AB	BC
bc	9	−5
ab	−6	9

→

	AB	BC
bc	10	−4
ab	−3	10

→

	AB	BC
bc	11	−1
ab	−2	11

Figure 2: **Sorting Need Not Rise in Synergy.** In the top row, the unique most efficient matchings alternates between NAM1 and NAM3. In the next row, all four match synergies — or the cross differences of match payoffs — strictly increase as we move right. So it is not try that increasing synergy leads to more sorting.

Legros and Newman (2002) noted that in the presence of imperfect credit constraints, supermodular production does not induce supermodular match payoff functions. Our nowhere decreasing theory subsumes their production function. But we instead focus on Guttman's (2008) dynamic extension of Ghatak's (1999) model of group lending with adverse selection — for which our stronger increasing sorting theory applies.

Another motivated twist on matching that undermines supermodularity is moral hazard. Serfes (2005) investigates a pairwise matching model of principals and agents. the more risk averse male is matched with the less risk averse female

Finally, even when static payoffs are supermodular, Anderson and Smith (2010) show that dynamic models with Bayesian updating need not inherit this. In our subsequent work with evolving human capital (Anderson and Smith, 2012), we show that preservation of supermodularity is highly exceptional. For general transition functions of old types into new types, the dynamic match values are rarely supermodular.

Becker's sorting result follow from standard monotone comparative statics results for supermodular functions (pursued at length in §3.2 in Topkis (1998)). Our insights hail from new results in monotone comparative statics, including two new ones. First, our nowhere decreasing theory owes to a comparative static result for partially ordered sets that are not lattices — which is the very character of bivariate matching distributions that obey adding up conditions. Lemma 1 summarizes our relevant findings here. Second, we derive an upcrossing aggregation result (Lemma 2) that extends the fundamental preservation result of Karlin and Rubin (1956) to vector domains.

Since we avoid standard optimization theory, multipliers play no role in the analysis and we thus are silent about wages.

Any results not proven immediately are demonstrated in the appendix.

2 The Marriage Model

A. The Marriage Model with Global Complements or Substitutes.

Assume pairwise matching by individuals either from two heterogenous groups (men and women, firms and workers, buyers and sellers) or the same set (partnerships). In the general matching model, "women" and "men" have respective *types* $x, y \in [0,1]$ with cdfs G and H. The matching market is balanced, with unit mass $G(1) = H(1) = 1$.

We capture a continuum of types by absolutely continuous type distributions G and H, and finitely many types when G and H are discrete measures with equal weights on male types $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$ and female types $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$. We then relabel women as $i = 1, 2, \ldots, n$ and men as $j = 1, 2, \ldots, n$.

We assume a C^2 *production function* $\phi > 0$, so that types x and y jointly produce $\phi(x, y)$. In the finite type model, the output for match (i, j) is $f_{ij} = \phi(x_i, y_j) \in \mathbb{R}$. Production is *supermodular* or *submodular* (SPM or SBM) for all $x' < x''$ and $y' < y''$ if:

$$\phi(x', y') + \phi(x'', y'') \geq (\leq)\ \phi(x', y'') + \phi(x'', y') \tag{2}$$

Strict supermodularity (respectively, strict SBM) asserts strict inequality in (2).

Like Becker's, our theory does not explore an extensive margin whether to match. A *matching* is a bivariate cdf $M \in \mathcal{M}(G, H)$ on $[0,1]^2$ with marginals G and H. In the finite type case, G and H put equal weight on $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. In this case, the matching is summarized by a nonnegative matrix $[m_{ij}]$ where $\sum_i m_{ij_0} = 1 = \sum_j m_{i_0 j}$ for all men i_0 and women j_0 in $\{1, 2, \ldots, n\}$. In a *pure matching*, $[m_{ij}]$ is a matrix of 0's and 1's: every man is matched to a unique woman, and vice versa.

Of longstanding interest are the two flavors of perfect sorting. In *positive assortative matching* (PAM), each woman x at quantile $G(x)$ pairs with a man y at the same quantile $H(y)$, and thus $M(x, y) = \min(G(x), H(y))$. In *negative assortative matching* (NAM), complementary quantiles match, and thus $M(x, y) = \max(G(x) + H(y) - 1, 0)$. Matched types are *uncorrelated* given uniform matching $M(x, y) = G(x)H(y)$.

Our matching model subsumes as a special case the *partnership* (or unisex) model, where types x and y share a common distribution, $G = H$, production ϕ is symmetric ($\phi(x, y) = \phi(y, x)$), and a symmetric matching distribution $M(x, y) \equiv M(y, x)$ is optimal. In this case, PAM is the matching $y = x$, and NAM the matching $y = 1 - x$.

The Planner seeks to maximize total match output, solving for optimal matchings:

$$\mathcal{M}^* = \arg \max_{M \in \mathcal{M}(G,H)} \int_{[0,1]^2} \phi(x, y) M(dx, dy) \tag{3}$$

Villani (2008) shows existence of $\mathcal{M}^*$ in his Theorem 4.1.[2] We prove uniqueness in §6.

Characterizing the solution of the optimization (3) is known as the *Transportation Problem*, and has been open since Monge (1781). Every feasible matching is optimal with modular production. Becker solved the extreme sorting cases PAM and NAM:

Becker's Theorem. *Given SPM (SBM) production ϕ, the optimal matching exists and is PAM (NAM). Given strict SPM (SBM), these pairings are uniquely optimal.*

Proof: Existence and uniqueness are by construction with finitely many types. For if any matches (x', y'') and (x'', y') are negatively sorted (so that $x' < x''$ and $y' < y''$), then output is not maximal, since SPM production (2) implies a higher payoff to the matches $(x', y') < (x'', y'')$. Lemma 1 addresses the general (non-finite) case.

This paper gives comparative statics for intermediate cases in which ϕ is neither SPM or SBM, and thus the optimal matching need not be at either extreme.

B. Matching with Local Complements and Local Substitutes.

Assume first finitely many types. Match *synergy* is the cross partial difference:

$$s_{ij} = f_{i+1j+1} + f_{ij} - f_{i+1j} - f_{ij+1}$$

Production is SPM, e.g., when synergy is globally nonnegative. The analysis is hard when synergy has mixed signs. We start by summing match output (3) by parts:

$$\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left[f_{nj+1} - f_{nj}\right] j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}M_{ij} \qquad (4)$$

So the production only impacts match output via synergy. So if production is linear in types, and synergy vanishes, then all match distributions yield the same output. Notice that in Becker's Theorem, synergy alone affects the optimal matching: *any two production functions with the same synergies have the same matching, if all match.*

In the continuum case, *synergy* is still a local complementarity notion: $\phi_{12}(x, y)$. The generalization of (4) to the continuum must carefully treat any type atoms.

Lemma 1 (Match Output Reformulated)**.** *Let $\mathcal{I} \equiv [0, 1]$ and $\mathcal{J} \equiv (0, 1]$. Then:*

$$\textstyle\int_{\mathcal{I}^2} \phi(x, y)M(dx, dy) = \int_{\mathcal{I}} \phi(x, 1)G(dx) - \int_{\mathcal{J}} \phi_2(1, y)H(y)dy + \int_{\mathcal{J}^2} \phi_{12}(x, y)M(x, y)dxdy$$

We don't solve the optimization (3), and instead index output $\phi(x, y|\theta)$ by a *state* θ in a partially ordered set (*poset*) Θ, and ask how optimizers $\mathcal{M}^*(\theta)$ change as θ increases. We often suppress the state. Throughout, a *time series* property relates production to the state, and a *cross-sectional property* relates production to the types.

[2] He claims it "has probably been known from time immemorial." Many papers have addressed this issue over the decades since Kantorovich (1942). See also Gretsky, Ostroy, and Zame (1992).

3 Economic Applications of the Marriage Model

We now explore some illustrative or celebrated economic applications of the marriage model not explained by Becker's Theorem — for production is neither SPM nor SBM.

A. Quadratic Production. We start with an instructive matching example. Since empirical work often ventures quadratic production, posit $\phi(x,y) = \alpha xy + \beta(xy)^2$. Then synergy $\phi_{12}(x,y) = \alpha + 4\beta xy$ is increasing in α and β. By Becker's Theorem, PAM is optimal when $\alpha, \beta \geq 0$, uniquely so if also $\alpha + \beta > 0$. Likewise, NAM is optimal when $\alpha, \beta \leq 0$, and uniquely so with $\alpha + \beta < 0$. But with either of $\alpha \lesseqgtr 0 \lesseqgtr \beta$, SPM and SBM fail, as synergy can be positive and negative; Becker's Theorem is inapplicable.

B. Principal-Agent Matching with Moral Hazard. Serfes (2005) explores a pairwise matching model of principals and agents. Project variances $y \in [\underline{y}, \overline{y}]$ vary across principals, while agents differ by their risk aversion parameter $x \in [\underline{x}, \overline{x}]$.

When agents share a common scalar dis-utility of effort $\theta > 0$, Serfes derives (in his equation (1)) the expected output and synergy of an (x, y) match:

$$\phi(x,y) = \frac{1}{2\theta\,(1+\theta xy)} \quad \Rightarrow \quad \phi_{12}(x,y) = \frac{\theta xy - 1}{2\,(1+\theta xy)^3} \tag{5}$$

Serfes applies Becker's Theorem to deduce NAM for $\theta < \underline{\theta}$ and PAM for $\theta > \overline{\theta}$. But he is silent about all intermediate disutility of efforts, where $1/[\overline{y}\,\overline{x}] = \underline{\theta} < \overline{\theta} = 1/[\underline{y}\underline{x}]$.

C. Group Lending with Adverse Selection. We consider Guttman's (2008) dynamic extension of Ghatak's (1999) model of group lending with adverse selection. Borrowers vary by their project success chance x; a success pays π and a failure nothing. Pairs of borrowers sign lending contracts, and project outcomes are independent.

After seeing the project outcome, a borrower either repays the loan, or defaults. If both repay, then each pays $r > 1$. But if only one defaults, then the other repays $r + c > r$. Assume $\pi \geq r + c$, so that borrowers repay when their project succeeds. If both matched borrowers default at once, each loses access to credit markets and can no longer finance projects. Borrowers discount future payoffs by $\delta < 1$, and default if the project fails. The discounted payoff to the matched success chance pair (x, y) obeys:

$$\phi(x,y) = x((\pi - r) - (1-y)c) + y((\pi - r) - (1-x)c) + \delta(1 - (1-x)(1-y))\phi(x,y) \tag{6}$$

One can check that synergy ϕ_{12} is globally positive if $\delta \leq \delta^* \equiv c/[c + (\pi - r)]$. But with more patience, $\delta > \delta^*$, synergy is positive for low (x, y) and negative for high (x, y).[3]

[3] Legros and Newman (2002) explore group borrowing to finance a *joint project*. In their model,

D. A Partnership Model with Capital. Modify the partnership model with types (x, y) so that the *effective labor* is $\ell(x, y) = (x^\eta + y^\eta)^{1/\eta}$. Assume that production also depends on the current technology (captured here by an index that might be viewed as 'capital' κ), match output and synergy:

$$\phi(x, y) = (\ell(x, y)^\rho + \kappa^\rho)^{1/\rho} \quad \Rightarrow \quad \phi_{12}(x, y) \propto (\rho - \eta)\kappa^\rho + (1 - \eta)\ell(x, y)^\rho \tag{7}$$

Assume greater complementarity between partner types than between labor and technology, $\rho < \eta < 1$. Then synergy is negative $\phi_{12} < 0$ for low types x, y, and positive for high types when $\rho > 0$. But if instead, $\rho < 0$, then synergy is positive for low types and negative for high types. In either case, Becker's Theorem does not apply.

E. Production with Defined Roles. Kremer and Maskin (1996) explore a unisex model with output equal to the maximum of two SPM functions:

$$\phi(x, y) \equiv \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\} \quad \text{for } \theta \in [0, 1/2] \tag{8}$$

Intuitively, assume that agents can be assigned to the manager or deputy roles, where $x^\theta y^{1-\theta}$ is output when x is assigned to the manager and y to the deputy.[4] This production function is neither SPM nor SBM. Indeed, consider any match (x, y) for $0 < x < y$. If $z = y/x$, the payoff difference of positive sorting minus negative sorting is:

$$\phi(y, y) + \phi(x, x) - 2\phi(x, y) = y + zy - 2(zy)^\theta y^{1-\theta} \gtreqless 0 \quad \text{as } \theta \gtreqless \theta^*(z)$$

where $\theta^*(z) = (\log(1 + z) - \log(2))/\log(z)$ is an increasing function from $(0, 1)$ onto $(0, 1/2)$. That is, PAM beats NAM among the types $\{x, y\}$ when types are far apart (small z), while NAM beats PAM when types close together (z near 1).

F. Dynamic Matching with Evolving Types. Assume pairwise matching in periods one and two. Production is the symmetric, increasing and SPM function $\phi^0(x, y)$. But types evolve: If types x and y match in period one, they enter period two as types $\tau(x, y)$ and $\tau(y, x)$, where τ is an increasing type transmission function.

Given SPM output, PAM is statically optimal in period two. But in period one, the social planner weights output by $(1 - \delta, \delta)$, so that the payoff to an (x, y) match is:

$$\phi(x, y) = (1 - \delta)\phi^0(x, y) + \delta\left[\phi^0(\tau(x, y), \tau(y, x)) + \phi^0(\tau(y, x), \tau(y, x))\right]/2$$

Becker's Theorem lacks bite: ϕ need not be SPM if ϕ^0 and τ are increasing and SPM.

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expected output is $\phi = (xy - q)\mathbb{1}_{XY \geq \kappa}$, where q is the cost of capital and $\kappa \geq q$ a financing constraint. Output is globally SPM when $\kappa = q$, but is neither globally SPM nor globally SBM when $\kappa > q$.

[4]More generally, we could allow for the production $\max\{g(x, y|\theta), g(y, x|\theta)\}$.

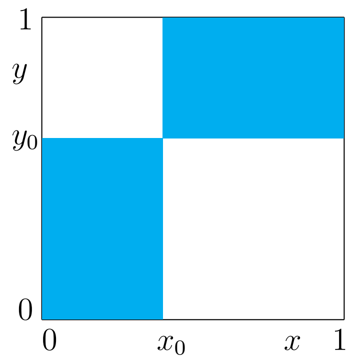


Figure 3: **The PQD Order.** PQD increases for cdfs on $[0,1]^2$ increase the probability mass on lower left rectangles with corner vertices $(0,0)$ and (x_0,y_0), and thus upper right rectangle with corner vertices (x_0,y_0) and $(1,1)$, for every $(x_0,y_0)\in[0,1]^2$.

4 A Measure of Sorting

Positive quadrant dependence (PQD) partially orders bivariate probability distributions $M_1, M_2 \in \mathcal{M}(G,H)$. We call M_2 *PQD higher than* M_1, or $M_2 \succeq_{PQD} M_1$, if $M_2(x,y) \geq M_1(x,y)$ for all x,y. So M_2 puts more weight than M_1 on all lower (southwest) orthants. Since M_1 and M_2 share marginals, M_2 puts more weight than M_1 on all upper (northeast) orthants too. Easily, the PQD order correctly ranks PAM, NAM and uniform matching: $\min(G(x),H(y)) \geq G(x)H(y) \geq \max(G(x)+H(y)-1,0)$. As noted in (1), this only partially orders the six possible pure matchings on three types.

The PQD and SPM orders coincide, i.e. for all SPM functions ϕ, since increases in the PQD order increase (reduce) the total output for any SPM (SBM) function:[5]

$$M_2 \succeq_{PQD} M_1 \quad \Leftrightarrow \quad \int \phi(x,y) M_2(dx,dy) \geq \int \phi(x,y) M_1(dx,dy) \tag{9}$$

We see now that Becker's Theorem follows from Lemma 1. For since SPM implies globally nonnegative synergy, $\phi_{xy} \geq 0$, output is highest when the match cdf $M(x,y)$ is maximal — namely, PAM, as it dominates all other matchings in the PQD order. Similarly, SBM implies globally nonpositive synergy, $\phi_{xy} \leq 0$, and thus output is highest when the match cdf $M(x,y)$ is minimal, namely, NAM.

The PQD sorting measure shows up in some economically relevant measures:

Lemma 2. *Fix increasing functions u and v. Given a PQD order upward shift:*
(a) the average geometric distance $E[|u(X)-v(Y)|^\gamma]$ for matched types falls, if $\gamma \geq 1$;
(b) the covariance $E_M[u(X)v(Y)] - E[u(X)]E[v(Y)]$ across matched pairs rises;
(c) the coefficient in a linear regression of $v(y)$ on $u(x)$ across matched pairs rises.

[5] Lehmann (1973) introduced the PQD order. See 9.A.17 in Shaked and Shanthikumar (2007).

Lemma 2 illustrates that the PQD order is scale invariant.To wit, if we claim that educational sorting rises in the PQD order, then it does so regardless of whether it is measured in highest degree attained, years of schooling, log years of schooling, etc.

Proof of (a): By inequality (9) it suffices that $|u(x) - v(y)|^\gamma$ is SBM for all $\gamma \geq 1$. Since $-g(u-v)$ is SPM for all convex g, by Lemma 2.6.2-(b) in Topkis (1998), we have $-|u-v|^\gamma$ SPM for all $\gamma \geq 1$. Thus, $|u(x) - v(y)|^\gamma$ is SBM for all increasing u and v.

Proof of (b): Since the marginal distributions on X and Y is constant for all $M \in \mathcal{M}(G, H)$, and $u(x)v(y)$ is supermodular for all increasing u and v, the covariance $E_M[XY] - E[X]E[Y]$ between matched types increases in the PQD order by (9).

Proof of (c): The coefficient $c_1 = cov(u(X)v(Y))/var(v(X))$ in the univariate match partner regression $v(y) = c_0 + c_1 u(x)$ increases in the PQD order, by part (b). □

For some helpful insight, assume a uniform distribution of types on $[0, 1]$, and assume that every $x \leq 1/2$ matches with $x + 1/2$. While Legros and Newman (2002) call this matching "monotone", because it is increasing *on the domain of larger match partners*. But this is not positive sorting since one can verify that it actually maximizes, rather than minimizes, the average distance between partners. To wit, it *minimizes* total match output for the supermodular production function $f(x, y) = 1 - |x - y|$.

5 Nowhere Decreasing Sorting

When the optimal matching $M^*(\theta)$ is unique, we say that *sorting is increasing* if it weakly increases in the PQD order. Figure 2 defeats a natural conjecture that sorting increases if synergy globally increases. For it shows that the uniquely optimal matching can shift back and forth between NAM1 and NAM3 as synergy strictly increases. But NAM1 and NAM3 are not PQD comparable by (1), and so sorting at least never falls.

To check whether this observation is generally true, we now index the production function $\phi(\cdot|\phi)$ by some parameter θ, and ask what happens to sorting as θ increases. Assume that synergy $\phi_{12}(\cdot|\phi)$ is non-decreasing in θ. By Lemma 1, the change in total output from the cdf M to M' is $\int \phi_{12}(\cdot|\theta)(M' - M)$. This gives a *single crossing condition* in (M, θ), i.e. for all $M' \succeq_{PQD} M$ (and thus $M' \geq M$ everywhere) and $\theta' \succeq \theta$:

$$\int_{(0,1]^2} \phi_{12}(\cdot|\theta)(M' - M) \geq (>)\ 0 \ \Rightarrow\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta')(M' - M) \geq (>)\ 0 \qquad (10)$$

A single crossing condition alone does not suffice for monotone comparative statics. If total output (3) were quasi-supermodular in M, and the constraint set a lattice, then (10) would imply that the set of maximizers $\mathcal{M}^*(\theta)$ increases in the *strong set order* (SSO) (result 2.8.6 in Topkis (1998)). Specifically, $\mathcal{M}_2 \succeq \mathcal{M}_1$ in the SSO if

$M_1 \vee M_2 \in \mathcal{M}_2$ and $M_1 \wedge M_2 \in \mathcal{M}_1$ for all $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$.[6] But sorting cannot rise in the SSO, since matching cdf's are not a lattice.[7]

Theorem 1 in §B.1 presents a theory of monotone comparative statics on partially ordered sets given only a single crossing property in (M, θ). This does not preclude PQD incomparable shifts of optimal matchings as θ rises, but it precludes declines. We say that *sorting is nowhere decreasing* in θ if for all $\theta_2 \succeq \theta_1$, whenever $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$ are ranked $M_1 \succeq_{PQD} M_2$, we have $M_2 \in \mathcal{M}^*(\theta_1)$ and $M_1 \in \mathcal{M}^*(\theta_2)$.

Lemma 3. *If total output* (3) *is single crossing in* (M, θ)*, then* $\mathcal{M}^*(\theta)$ *is nowhere decreasing in* θ*. Conversely, if* $\mathcal{M}^*(\theta)$ *is nowhere decreasing in* θ *for all type distributions* G, H*, then total output is single crossing in* (M, θ)*.*

We say that *weighted synergy is upcrossing* in θ if the following is upcrossing in θ: $\int \phi_{12}(x, y|\theta)\lambda(dx, dy)$ with continuous types and C^2 production, for all positive measures λ on $[0, 1]^2$, and $\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}\lambda_{ij}$ with finite types, for all weights $\lambda \in \mathbb{R}_+^{(n-1)^2}$.

Proposition 1 (Nowhere Decreasing Sorting)**.** *Sorting is nowhere decreasing in* θ *if weighted synergy is upcrossing in* θ *— and so if synergy is non-decreasing in* θ*.*

Proof: First, $M' \succeq_{PQD} M$ iff $\lambda \equiv M' - M \geq 0$. So if weighted synergy is upcrossing in θ, total output obeys the single crossing condition (10) for continuous types, and an analogous one with finite types. By Lemma 3, sorting is nowhere decreasing in θ. □

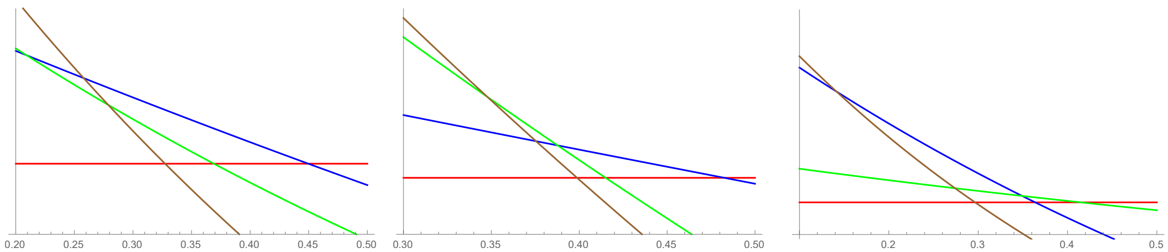
Application: Production with Defined-roles. Now return to Kremer and Maskin (1996) in §3. Their production function is not differentiable, and so invalidates our theory. But we can consider smoothly approximate their production function by:

$$\phi(x, y) = x^\theta y^\theta \left(x^\varrho + y^\rho\right)^{\frac{1-2\theta}{\varrho}} \quad \to \quad \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\} \quad \text{as } \varrho \to \infty \qquad (11)$$

One can check that ϕ is SPM iff $\varrho < (1 - 2\theta)^{-1}$, and so PAM arises, by Becker's Theorem. In the $\varrho \to \infty$ limit of Kremer and Maskin (1996), ϕ is never SPM, and PAM does not arise. Even though we cannot possibly solve for the optimal matching when $\varrho \geq (1 - 2\theta)^{-1}$, our theory affords signs of the sorting comparative statics. In Appendix D, we prove that synergy is upcrossing in θ and downcrossing in ϱ, and thus *sorting is nowhere-decreasing in* θ *and* $1/\varrho$. Figure 4 plots total match payoffs for $f(x, y) = \max(x^\theta y^{1-\theta}, x^{1-\theta} y^\theta)$, with three types $x, y \in \{1, L, H\}$, for various $H > L > 1$.

[6] Given M_1, M_2, the *join* $M_1 \vee M_2$ is their supremum and the *meet* $M_1 \wedge M_2$ their infimum.

[7] By (1), NAM1 and NAM3 are both upper bounds for PAM2 and PAM4, but there is no pure least upper bound. More strongly, PQD does not induce a lattice, as there is no least mixed least upper bound, M for PAM2 and PAM4. As shown in Proposition 4.12 in Müller and Scarsini (2006): If M dominates PAM2 and PAM4, then $M(2, 1) \geq 1/3$ and $M(1, 2) \geq 1/3$, but $M(1, 1) = 0$ if NAM1 and NAM3 dominate M. So then $M(2, 2) = 2/3$, but then NAM1 cannot PQD dominate M.



production parameter θ. production parameter θ. production parameter θ.

Figure 4: **Kremer-Maskin Payoffs.** We plot payoffs for the four pure symmetric three type matchings against θ. NAM (brown) is optimal for low θ and PAM (red) for high θ. Sorting increases in θ at left with NAM1 (blue) optimal for intermediate values of θ. In the middle and right, sorting is not monotone, as the PQD-incomparable NAM1 (blue) and NAM3 (green) are each optimal for a range of parameter values.

6 Increasing Sorting and Production Changes

6.1 Increasing Sorting and the Sorting Premium Theory

We first focus on the finite case with female and male types $i, j = 1, 2, \ldots, n$.

Lemma 4. *An optimal matching is generically unique and pure for finite types.*

Proof: The optimal matching is generically unique, by Koopmans and Beckmann (1957). Since any non-pure matching M is a mixture $M = \sum_{k=1}^{K} \lambda_k M_k$ over $K \leq n+1$ pure matchings $M_1, \ldots, M_n$, with $\lambda_k > 0$ and $\sum_k \lambda_k = 1$.[8] As the objective function (3) is linear, if the non-pure matching M is optimal, so is each pure matching M_k. □

We now introduce a cross-sectional assumption on how synergy changes across types. At the core of our theory is the *sorting premium* defined on rectangles $R = (x_1, y_1, x_2, y_2)$ in type space with diagonally opposite vertices $(x_1, y_1) < (x_2, y_2)$:

$$S(R|\theta) \equiv \phi(x_1, y_1|\theta) + \phi(x_2, y_2|\theta) - \phi(x_1, y_2|\theta) - \phi(x_2, y_1|\theta)$$

By (2), production is SPM (SBM) if all sorting premia are nonnegative (non-positive).[9]

Rectangle R dominates R', written $R \succeq R'$, if all coordinates are weakly higher and $R \succ R'$ if at least one coordinate is strictly higher. The sorting premium is *upcrossing (downcrossing) in types* if $S(R|\theta)$ is upcrossing (downcrossing) in R, for all θ. Since we can always reverse-order types, we just develop our theory for the upcrossing case.

While sorting is nowhere decreasing in synergy, by Figure 2, it is not increasing.

[8] This follows from Carathéodory's Theorem. It says that non-empty convex compact subset $\mathcal{X} \subset \mathbb{R}^n$ are weighted averages of extreme points of $\mathcal{X}$. The extreme points here are the pure matchings.

[9] By Proposition 1, sorting is nowhere decreasing if weighted synergy is upcrossing in θ. The sorting premium uses a weighting function that places unit density on a rectangle, and zero weight elsewhere.

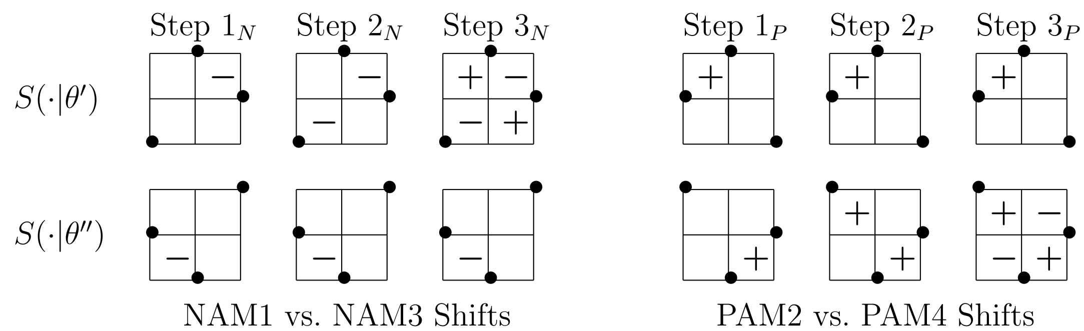


Figure 5: **Precluding Unranked Shifts with $n = 3$ and Nonzero Synergies.** NAM1 at θ' and NAM3 at θ'' is impossible, as is PAM2 at θ' and PAM4 at θ''. The synergy signs in Steps 1_N and 1_P reflect local optimality. Step 2_N deduces $s_{11}(\theta') < 0$ via upcrossing synergy from θ'' to θ'. Given PAM on rectangles $R = (1, 1|2, 3), (1, 1|3, 2)$ at θ', local optimality implies $S(R|\theta') > 0$. As the sorting premium is the sum of synergies, the synergy signs in Step 3_N follow — ruling out $S(R|\theta')$ one-crossing in R, a contradiction. Next, Step 2_P deduces $s_{12}(\theta'') > 0$ via upcrossing synergy from θ' to θ''. Given NAM on rectangles $R = (1, 1|2, 3), (1, 1|3, 2)$ at θ', local optimality implies $S(R|\theta') < 0$. Since the sorting premium is the sum of synergies, we can fully sign s_{ij}. This sign pattern in Step 3_P violates $S(R|\theta'')$ one-crossing in R, a contradiction.

Proposition 2. *For the generically unique production functions with finitely many types, sorting is increasing in θ, if $S(R|\theta)$ is upcrossing in θ and one-crossing in R.*

In Appendix C.2, we prove this by induction on the number n of types. Here, we sketch out the proof logic for the three type examples in Figure 1 in which monotonicity fails. We assume the upcrossing case throughout, for definiteness — but this is WLOG, as we can inversely order types.

Consider a generic case with a unique and pure optimal matchings M' and M'', by Lemma 5 for states $\theta'' \succeq \theta'$. We first rule out $M' \succ_{PQD} M''$, say, $M' =$ PAM4 and $M'' =$ NAM.[10] Since PAM4 includes a PAM pair on the lower right quadrant, the sorting premium obeys $S(R_4|\theta') \geq 0$. Since S is upcrossing in θ, we have $S(R_4|\theta'') \geq 0$. But then NAM cannot be uniquely optimal for θ'', since NAM and PAM4 differ on R_4.

The result follows if we rule out M' and M'' incomparable when $S(R|\theta)$ is upcrossing in R and θ. By the partial order (1), we must rule out transitions between NAM1 and NAM3, and between PAM2 and PAM4. Figure 5 argues these cases.

Trading Application. Assume n potential sellers j of houses with "costs" $c_1 < \cdots < c_n$, and n potential buyers i with valuations $v_{i1} > \cdots > v_{in}$ (Shapley and Shubik, 1971). The value of match (i, j) is thus $f(i, j) = \max(v_{ij} - c_j, 0)$. Assume that $\theta \geq 0$ increases the buyers values . . . (to be continued) . . .

[10]Proposition 1 does not rule out a PAM4 to NAM fall, as it requires upcrossing weighted synergy for *any* positive weighting function, while Proposition 3 only assumes uniform weights on rectangles.

6.2 Increasing Sorting and the Marginal Product Increment

We now reformulate our cross-sectional premise of Proposition 2. Call $\Delta_x(x|y_1, y_2, \theta) \equiv \phi_1(x, y_2|\theta) - \phi_1(x, y_1|\theta)$ and $\Delta_y(y|x_1, x_2, \theta) \equiv \phi_2(x_2, y|\theta) - \phi_2(x_1, y|\theta)$ the *x-marginal product increment* and *y-marginal product increment*, respectively. As production is C^2, we can compute the sorting premium from either marginal product increment:

$$S(R|\theta) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi_{12}(x, y|\theta) dx dy = \int_{y_1}^{y_2} \Delta_y(y|x_1, x_2, \theta) dy = \int_{x_1}^{x_2} \Delta_x(x|y_1, y_2, \theta) dx \quad (12)$$

Lemma 5 (Uniqueness)**.** *An optimal matching is unique given an absolutely continuous cdf G (or H) and the x- (or y-) marginal product increment strictly one-crossing.*[11]

Our continuum uniqueness proof in Appendix C.3 applies Theorem 5.1 in Ahmad, Kim, and McCann (2011). Here we offer a novel intuition for why absolutely continuous G and $\Delta_x(x|y_1, y_2)$ strictly upcrossing in x give uniqueness. Any optimal matching can be decentralized by competitive wage functions $v(x)$ and $w(y)$, where x and y match if $x = \arg\max_{x'} \phi(x', y) - v(x')$ and $y' = \max_{y'} \phi(x, y') - w(y')$. Assume two women $x_1 < x_2$ and two men $y_1 < y_2$. We can argue why both positive sorting (x_1, y_1) and (x_2, y_2) and negative sorting (x_1, y_2) and (x_2, y_1) cannot be optimal. For if so, since G is absolutely continuous and production ϕ is C^2, an optimal partner for y obeys the first order conditions: But $v'(x_1) = \phi_1(x_1, y_2) = \phi_1(x_1, y_1)$ and $v'(x_2) = \phi_1(x_2, y_1) = \phi_1(x_2, y_2)$ contradicts $\Delta_x(x|y_2, y_1) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x.

Proposition 3 (Increasing Sorting, II)**.** *Sorting is increasing in θ if either cdf G or H is absolutely continuous, the sorting premium $S(R|\theta)$ is strictly upcrossing in θ, and the x- and y-marginal product increments are both strictly one-crossing in the same direction.*

In Appendix C.2, we prove this by appeal to upper hemicontinuity and uniqueness logic as we near the continuum type distribution. The above premise implies the premise of Lemma 5, and so the optimum is unique. It follows from Proposition 2 provided the sorting premium $S(R|\theta)$ is one-crossing in R. Rewrite equation (12) as:

$$S(R|\theta) = \int \Delta_x(x|y_1, y_2, \theta) \mathbb{1}_{x \in [x_1, x_2]} dx$$

Since the indicator function $\mathbb{1}_{x \in [x_1, x_2]}$ is log-SPM in (x, x_1) and in (x, x_2),[12] $S(R|\theta)$ is upcrossing in x_1 and in x_2, and thus in (x_1, x_2), whenever $\Delta_x(x|y_1, y_2, \theta)$ is upcrossing in x,

[11] We call any function $\Upsilon : \mathbb{R} \mapsto \mathbb{R}$ *strictly upcrossing* if, for all $x' > x$, we have $\Upsilon(x) \geq 0 \Rightarrow \Upsilon(x') > 0$. Easily, a strictly upcrossing function is upcrossing.

[12] It suffices to check that if $x \in [x_1, x_2]$ and $x' \in [x'_1, x'_2]$ then $\max(x, x') \in [\max(x_1, x'_1), \max(x_2, x'_2)]$ and $\min(x, x') \in [\min(x_1, x'_1), \min(x_2, x'_2)]$.

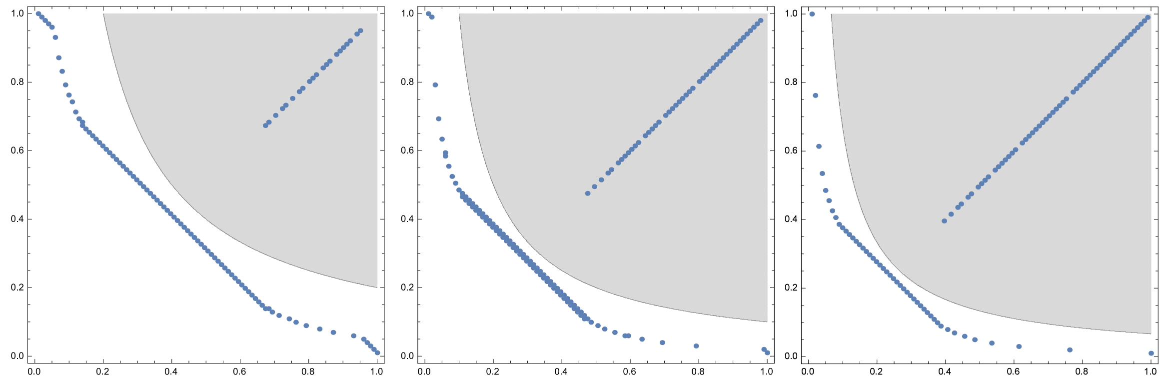


Figure 6: **Increasing Sorting in the Principal-Agent Model.** NAM is optimal for low dis-utility of effort θ, PAM for high θ, and the optimal matching is mixed for intermediate θ. These graphs depict optimally matched pairs (blue dots) assuming a discrete uniform distribution on 100 types of principals and agents. The left matching is drawn for $\theta = 5$, the middle for $\theta = 10$, and the right is for $\theta = 15$.

by Karlin and Rubin (1956). Likewise, if $\Delta_y(y|x_1, x_2, \theta)$ upcrossing in y, then $S(R|\theta)$ is upcrossing in (y_1, y_2). Then $S(R|\theta)$ is upcrossing in $R = (x_1, y_1, x_2, y_2)$ when the x- and y- marginal product increments are upcrossing. Claim C.6 proves that $S(R|\theta)$ is *strictly* upcrossing in R given strictly upcrossing marginal product increments.

We now return to two of the economic applications in §3.

Application to the Principal-Agent Matching with Moral Hazard. We verify that Serfes' production function (5) obeys the premise of Proposition 3. Indeed:

$$\Delta_x(x|y_1, y_2, \theta) = y_1 - y_2 \left(\frac{1 + \theta x y_1}{1 + \theta x y_2}\right)^2$$

is strictly increasing in x and θ, since $y_2 > y_1$. By symmetry, the y-marginal product increment Δ_y is strictly upcrossing in y and θ. Also, since $\Delta_x(x|y_1, y_2, \theta)$ is strictly increasing in θ, the sorting premium $S(R|\theta) = \int \Delta_x(x|y_1, y_2, \theta)\mathbb{1}_{x \in [x_1, x_2]}$, is strictly increasing (and so strictly upcrossing) in θ. Hence, sorting is increasing in θ. Figure 6 plots the solution for increasing values of θ (left to right).[13]

Application to Group Lending with Adverse Selection. The production function obeys the premise of Proposition 3. Differentiating (6) yields:

$$\phi_1(x, y) = \frac{(\pi - r - c)(1 - \delta y^2) + 2cy}{(1 - \delta x - \delta y + \delta x y)^2} > 0$$

Then $\partial[\phi_1(x, y_2)/\phi_1(x, y_1)]/\partial x < 0$ for all $y_2 > y_1$, and thus, the x-marginal product

[13] These finite plots suggest that the unique continuum matching is not always pure, but fortunately, none of our continuum model sorting results require purity.

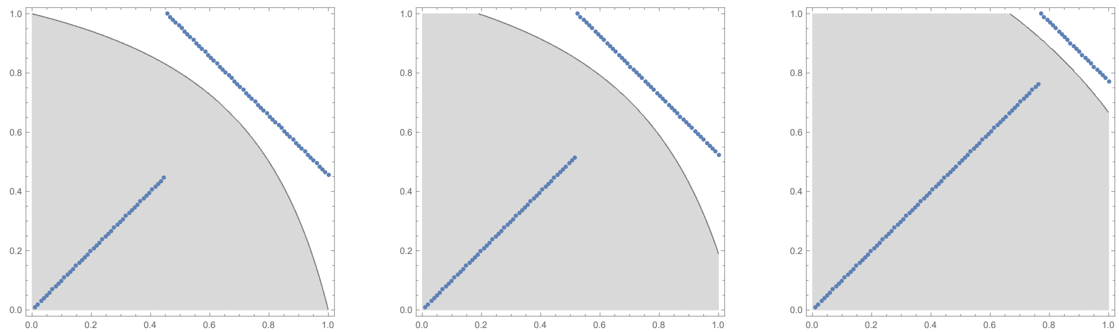


Figure 7: **Increasing Sorting in the Group Lending Model.** For $\delta < c/(c+\pi-r)$, the optimal is PAM, above this threshold the optimal matching is mixed. These graphs depict optimally matched man-woman pairs (blue dots) assuming a uniform distribution on 100 types (blue dots). The left matching is drawn for $(\pi, r, c) = (8, 2, 0)$, the middle for $(\pi, r, c) = (8, 2, 1)$, and the right for $(\pi, r, c) = (3, 2, 1)$.

increment $\phi_1(x, y_2) - \phi_1(x, y_1)$ is strictly downcrossing in x. Symmetrically, the y-marginal product increment is strictly downcrossing in y. Next, write synergy as

$$\phi_{12}(x,y) = \frac{2[c(1-\delta)(1-\delta xy) + \delta(\pi - r)(1-x-y+\delta xy)]}{(1-\delta x - \delta y + \delta xy)^3} \equiv cA(x,y) + (\pi - r)B(x,y)$$

where $A(x,y) > 0$, and thus, the sorting premium $S(R|\theta)$ in (12) is strictly increasing in c. It is also strictly downcrossing in $\theta = \pi - r$, since for all $\theta'' > \theta'$:

$$S(R|\theta') \leq 0 \quad \Leftrightarrow \quad c\int_R A(x,y) + \theta' \int_R B(x,y) \leq 0 \;\Rightarrow\; c\int_R A(x,y) + \theta'' \int_R B(x,y) < 0$$

Altogether, sorting is increasing in the repayment amounts (r, c) and decreasing in the payoff from a success π by Proposition 3, as illustrated in Figure 7.

Sorting is not monotone in the discount factor δ. PAM obtains with sufficient impatience $\delta \leq c/(c+\pi-r)$ *and* perfect patience $\delta = 1$,[14] since the sorting premium is globally positive in these cases. For intermediate $\delta \in (c/(c+\pi-r), 1)$, the sorting premium is not globally positive and PAM is not optimal in the continuum model.[15]

In these and other applications, verifying the upcrossing sorting premium premise is straightforward. But quite often, it is intractable. In the next section, we derive conditions on synergy that deliver this conclusion.

[14]Payoffs are well defined when the implicit discount factor $\delta(1-(1-x)(1-y)) < 1$, where $(1-x)(1-y)$ is the chance that both projects fail, resulting in the borrowing partnership defaulting.

[15]Indeed, when $\delta \in (c/(c+\pi-r), 1)$, the symmetric synergy function $\phi_{12}(x,x)$ is strictly negative for x close to 1. Thus, cross matching types x and $x+\varepsilon$ beats sorting them, for high x and low ε.

6.3 Increasing Sorting and Synergy

For our second differential variation on the cross-sectional assumption of Proposition 2, notice that for very small rectangles, upcrossing synergy is *necessary* for an upcrossing sorting premium. We now ask when it is also *sufficient* — namely, under what other cross sectional or time series condition is upcrossing synergy preserved by integration.

Our theory exploits a new upcrossing aggregation result in §B.2.[16] Consider the synergy σ, either $\sigma(x,y) = \phi_{12}(x,y)$ on the lattice domain $\mathcal{D} = [0,1]^2$, or $\sigma(i,j) = s_{ij}$ on $\mathcal{D} = \{1,\ldots,n\}^2$. Synergy is *proportionately upcrossing* if for all $\theta' \succeq \theta$ and $z, z' \in \mathcal{D}$, we have:[17]

$$\sigma^-(z \wedge z', \theta)\sigma^+(z \vee z', \theta') \geq \sigma^-(z, \theta')\sigma^+(z', \theta) \tag{13}$$

We analogously define *proportionately downcrossing* and *proportionately one-crossing*, except that the former assumes $\theta' \succeq \theta$, so that higher θ still denotes more sorting. Synergy is *proportionately upcrossing / downcrossing in types* if (13) holds for fixed θ.

For example, first consider the latter cross-sectional property. With three types, synergies $s_{11} = -1, s_{12} = 2, s_{21} = -4, s_{22} = 3$ are strictly upcrossing in i and j. But the associated sorting premium is not upcrossing in types — for instance, $s_{11} + s_{12} = 1 > -1 = s_{21} + s_{22}$. And indeed, it is not proportionately upcrossing, since at $z = (2,1)$ and $z' = (1,2)$, we have $s^-_{z\wedge z'} s^+_{z\vee z'} = 3 < 8 = s^-_z s^+_{z'}$. Intuitively, the proportionate gain in negative synergy $s_{21}/s_{11} = 4$ swamps that in positive synergy $s_{22}/s_{12} = 3/2$.

Next, consider the time series implications of inequality (13). Easily, any monotonic synergy function obeys it, since $(z', \theta) \leq (z \vee z', \theta')$ implies $\sigma^+(z \vee z', \theta') \geq \sigma^+(z', \theta)$, and $(z \wedge z', \theta) \leq (z, \theta')$ implies $\sigma^-(z \wedge z', \theta) \geq \sigma^-(z, \theta')$. Yet monotonicity is not implied — synergy function need only be *weakly upcrossing* in (z, θ); namely, $\sigma(z, \theta) > 0$ implies $\sigma(z', \theta') \geq 0$ for all $(z', \theta') \geq (z, \theta)$.[18] To see how this strengthens weakly upcrossing, assume negative synergy at couple z, and positive at a higher couple $z' = z \vee z' \geq z \vee z' = z$. Then (13) says that positive synergy proportionately rises more than the negative synergy rises, or falls proportionately less than negative synergy falls: $\sigma^+(z', \theta')/\sigma^-(z, \theta') \geq \sigma^+(z', \theta)/\sigma^-(z, \theta)$.

Lemma 6 (Aggregation)**.** *Assume proportionately one-crossing synergy.*
(a) Then $S(R|\theta)$ is generically upcrossing in states and one-crossing in types;
(b) If synergy is strictly upcrossing in states and strictly one-crossing in types, the x- and y-marginal product increments are both strictly one-crossing in the same direction.

We are now positioned to apply Propositions 2 and 3.

[16] We relate this condition and "signed-ratio monotonicity" (Quah and Strulovici, 2012) in §B.2.

[17] Denote by $f^+ \equiv \max(f, 0)$ and $f^- \equiv -\min(f, 0)$ the positive and negative parts of a function f.

[18] Fix $\theta = \theta'$ and suppress θ. If $z' \geq z$, inequality (13) is an identity. If $z' < z$, inequality (13) becomes $\sigma^-(z')\sigma^+(z) \geq \sigma^-(z)\sigma^+(z')$, which precludes $\sigma(z) < 0 < \sigma(z')$.

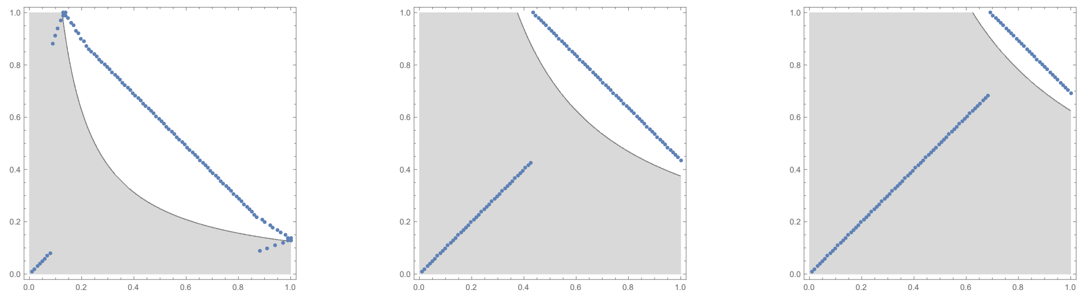


Figure 8: **Increasing Sorting in the Quadratic Model.** These graphs depict optimally matched pairs (blue dots) assuming a uniform distribution on 100 types. Sorting increases in (α, β) from left to right. The left matching is drawn for $(\alpha, \beta) = (0.5, -1)$, the middle for $(\alpha, \beta) = (1.5, -1)$, and the right for $(\alpha, \beta) = (1.5, -0.6)$.

Proposition 4 (Increasing Sorting, III)**.** *Assume proportionately one-crossing synergy. Sorting is increasing in θ in generic finite models, or if type distribution G (or H) is absolutely continuous and synergy is strictly upcrossing in θ and one-crossing in R.*

A simple corollary is that sorting is increasing in θ given finite types and monotone synergy, or with absolutely continuous G or H and strictly monotone synergy.

This result sheds light on two of the economic applications in §3.

Quadratic and Cubic Production Application. With quadratic production $\phi(x, y) = \alpha xy + \beta(xy)^2$, match synergy $\phi_{12} = \alpha + 4\beta xy$ is strictly increasing in α and β and strictly increasing (decreasing) in types when $\beta > 0$ ($\beta < 0$). Thus, sorting is increasing in α and β for all $\beta \neq 0$, as depicted in Figure 8.

With cubic production $\phi = \alpha xy + \beta(xy)^2 + \gamma(xy)^3$, the analysis is more nuanced. Synergy $\phi_{12} = \alpha + 4\beta xy + 9\gamma(xy)^2$ is increasing in α, β, and γ; and thus, sorting is nowhere decreasing in all parameters, by Proposition 1. Also, synergy falls in types when $\beta, \gamma < 0$, and rises in types when $\beta, \gamma > 0$ — so that Proposition 4 predicts sorting increases in α, β, and γ. But when $\beta\gamma < 0$, synergy need not be one-crossing in types, and sorting is nowhere decreasing, but not generally monotone in α, β, or γ.[19]

Application to the Partnership Model with Capital. We explore how changing technology (captured by capital κ) impacts sorting when $\rho < \eta < 1$, i.e. partners are more complementary than capital and labor. For the production function (7), synergy is:

$$\phi_{12}(x, y) = \varsigma(x, y|\kappa) \left[(\rho - \eta)\kappa^\rho + (1 - \eta)\ell(x, y)^\rho\right]$$

where $\varsigma(x, y|\kappa) \equiv (xy)^{\eta-1}\ell(x, y)^{\rho-2\eta}\phi(x, y)^{1-2\rho}$. We posit $2\eta \geq 1 - \rho$, so that ς is

[19]For example, we can construct three type examples in which the unique optimal matching shifts from NAM1 to NAM3 (or vice-versa) as the vector (α, β, γ) increases.

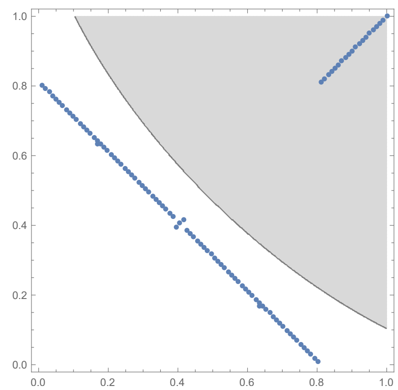


Figure 9: **Sorting in the Partnership Model with Capital.** [other plots?] We assume $k = 0.5, \rho = 0.3$, and $\eta = 0.7$. Positive synergy is greyed in. We depict the support of the optimal matching for a discrete uniform distribution on types (blue dots).

log-SPM in (x, y).

We claim that sorting increases in κ if $\rho < 0$. For $g(x, y|\kappa)$ falls in (x, y) and increases in κ. So synergy is strictly downcrossing in (x, y) and strictly upcrossing in κ. Also, ς is log-SPM in (x, y, κ) when $\rho < 0$.[20] So synergy is proportionately upcrossing and strictly upcrossing in $(-x, -y, \kappa)$, and sorting rises in κ, by Proposition 4.

In Appendix D.2, we show that sorting falls in the technology level κ.

7 Increasing Sorting and Type Distribution Shifts

We leverage our production comparative statics to deduce predictions for changes in the type distributions G_θ and H_θ. We say that X *types shift up (or down)* in θ if G_θ stochastically increases (or decreases) in θ, i.e. $G_{\theta'}(\cdot) < G_\theta(\cdot)$ if $\theta' \succ \theta$. Similarly, Y *types shift up (or down)* in θ if H_θ stochastically increases (or decreases) in θ, if $\theta' \succ \theta$.

The PQD sorting order only ranks matching distributions with the same marginals. For any matching cdf M', we therefore consider the associated bivariate *copula* $C(p, q) = M(X_\theta(p), Y_\theta(q))$, where $X_\theta(p) = G_\theta^{-1}(p)$ is the p *quantile* and $Y_\theta(q) = H_\theta^{-1}(q)$ is the q *quantile*. If the matching cdfs M' and M'' share the same marginals, then *quantile sorting increases* M' if their associated copulas to M', M'' are ranked $C'' \succeq_{PQD} C'$.

While the production function $\phi(x, y)$ no longer depends on θ, neither does the sorting premium $S(x_1, x_2, y_1, y_2)$ or finite- or continuous-type synergy. Nevertheless, the *quantile sorting premium* $\mathcal{S}(p_1, q_1, p_2, q_2|\theta) = S(X_\theta(p_1), Y_\theta(q_1), X_\theta(p_2), Y_\theta(q_2))$ does.

Proposition 5 (Types)**.** *Quantile sorting increases in θ if types shift up (down) in θ: (a) generically with finitely many types, if the sorting premium $S(R)$ upcrosses (downcrosses), and so if synergy is proportionately upcrossing (downcrossing) in types.*

[20]Since ς is log-SPM when $2\eta \geq 1 - \rho$, it is log-SPM when $\rho \geq 1/2$ and $\rho < 0$. Also, ς is log-SPM in (x, κ) and (y, κ) whenever $\rho(2\rho - 1) \geq 0$, i.e. $\rho \geq 1/2$ or $\rho \leq 0$.

(*b*) *given* G_θ, H_θ *absolutely continuous, if both marginal product increments strictly upcross (downcross), and so if synergy proportionately upcrosses (downcrosses) in types, and strictly upcrosses (downcrosses) in types.*

Proof Part (*a*): As types shift up in θ, quantiles $X_\theta(p)$ and $Y_\theta(q)$ increase in (p, q, θ), and so $\mathcal{S}(p_1, q_1, p_2, q_2|\theta) = S(X_\theta(p_1), Y_\theta(q_1), X_\theta(p_2), Y_\theta(q_2))$ upcrosses in (p_1, q_1, p_2, q_2) and θ if $S(x_1, y_1, x_2, y_2)$ is upcrossing. Quantile sorting increases in θ, by Proposition 2. Finally, by Lemma 6, the sorting premium $S(R)$ upcrosses (downcrosses) when synergy is proportionately upcrossing (downcrossing) in type.

Proof Part (*b*): The quantile $X_\theta(p)$ is increasing in p and θ given G_θ absolutely continuous and falling in θ. Given a strictly upcrossing marginal product increment $\Delta_x(x|y_1, y_2)$, the p-quantile marginal product increment

$$\Delta_p(p|q_1, q_2, \theta) = \Delta_x(X_\theta(p)|Y_\theta(q_1), Y_\theta(q_2))X'_\theta(p)$$

is strictly upcrossing in p. As H_θ absolutely continuous, the q-quantile marginal product increment $\Delta_q(q|p_1, p_2, \theta)$ is strictly upcrossing in q. Also, given $\Delta_x(x|y_1, y_2)$ and $\Delta_y(y|x_1, x_2)$ strictly upcrossing in x and y, respectively, the sorting premium $S(R|\theta)$ is *strictly* upcrossing in R, as shown in the proof of Proposition 3. Since $S(R|\theta)$ is strictly upcrossing in R, and $X_\theta(p)$ and $Y_\theta(q)$ are increasing in θ, $\mathcal{S}(p_1, q_1, p_2, q_2|\theta)$ is strictly upcrossing in θ. Altogether, quantile sorting rises in θ, by Proposition 3. Finally, by Lemma 6, if synergy strictly proportionately upcrosses (downcrosses) in types, then the x-and y- marginal product increments both strictly upcross (downcross). □

Applications to Earlier Examples. In §6, we established the one-crossing properties for our examples that meet the premise of Proposition 5: Namely, the x-and y- marginal product increments are strictly increasing in the principal-agent model; and so, quantile sorting increases when types shift up. In the group lending model, the x-and y- marginal product increments both strictly downcross: Quantile sorting increases when types shift down. In the quadratic production model, synergy is increasing when $\beta > 0$ (decreasing when $\beta < 0$); and thus synergy is proportionately upcrossing (downcrossing) in types, and quantile sorting rises when types shift up (down).

8 Conclusion

Becker's insight that supermodularity yields positive sorting sparked a huge literature on matching. But the incredible mathematical complexity has prevented any general theory for non-assortative matching. This has not stopped many impressive set of motivated models without perfect sorting. Still, the absence of a general theory has rendered all such progress impressively hard. This paper provides this missing general theory by sidestepping the solution of the optimal matching; we instead uses the logic of monotone comparative statics to answer when the matching grows more assortative.

Our paper subsumes existing influential work, but offers a set of easily checked conditions on changes in productivity or type distributions that deliver increased sorting. Our paper should prove useful for theoretical and empirical analysis.

A Match Output Reformulation: Proof of Lemma 1

Finite Types. Summing $\sum_{i=1}^{N}\left[\sum_{j=1}^{N} f(i,j|\theta)m_{ij}\right]$ by parts in j and then i yields:

$$
\begin{aligned}
& \sum_{i=1}^{N}\left[f(i,N)\sum_{j=1}^{N} m_{ij} - \sum_{j=1}^{N-1}[f(i,j+1|\theta)-f(i,j|\theta)]\sum_{k=1}^{j} m_{ik}\right] \\
= \; & \sum_{i=1}^{N} f(i,N) - \sum_{j=1}^{N-1}\sum_{i=1}^{N}[f(i,j+1|\theta)-f(i,j|\theta)]\sum_{k=1}^{j} m_{ik} \\
= \; & \sum_{i=1}^{N} f(i,N) - \sum_{j=1}^{N-1}\left([f(N,j+1|\theta)-f(N,j|\theta)]\sum_{\ell=1}^{N}\sum_{k=1}^{j} m_{\ell k} - \sum_{i=1}^{N-1} s_{ij}\sum_{\ell=1}^{i}\sum_{k=1}^{j} m_{\ell k}\right) \\
= \; & \sum_{i=1}^{N} f(i,N) - \sum_{j=1}^{N-1}\left([f(N,j+1|\theta)-f(N,j|\theta)]\, j - \sum_{i=1}^{N-1} s_{ij}M_{ij}\right)
\end{aligned}
$$

Continuum Types. If f is C^1 on $[0,1]$ and Γ is a cdf on $[0,1]$, integration by parts yields:

$$\int_{[0,1]} f(z)\Gamma(dz) = f(1)\Gamma(1) - \int_{(0,1]} f'(z)\Gamma(z)dz \tag{14}$$

where the interval $(0,1]$ accounts for the possibility that Γ may have a mass point at 0. Since $M(dx,y) \equiv M^Y(y|x)G(dx)$ for a conditional matching cdf $M^Y(y|x)$, we have:

$$M(x,y) \equiv \int_{[0,x]} M^Y(y|x')G(dx') \tag{15}$$

Applying Theorem 34.5 in Billingsley (1995) and then in sequence (14), (15) and Fubini, (14), the objective function $\int_{[0,1]^2} \phi(x,y)M(dx,dy)$ in (3) equals:

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,1]} \phi(x,y)M^Y(dy|x)G(dx) \\
=\;& \int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]}\int_{(0,1]} \phi_2(x,y)M^Y(y|x)dyG(dx) \\
=\;& \int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]}\int_{[0,1]} \phi_2(x,y)M(dx,y)dy \\
=\;& \int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]} \left[\phi_2(1,y)M(1,y) - \int_{(0,1]} \phi_{12}(x,y)M(x,y)dx\right] dy
\end{aligned}$$

which easily reduces to the expression in Lemma 1 using $M(1,y) = H(y)$.

B New Results in Monotone Comparative Statics

B.1 Nowhere Decreasing Optimizers

We now show that nowhere decreasing sorting is the appropriate partial order on sets of maximizers of single-crossing functions on partially ordered sets (*posets*).

Let Z and Θ be posets. The correspondence $\zeta : \Theta \to Z$ is *nowhere decreasing* if $z_1 \in \zeta(\theta_1)$ and $z_2 \in \zeta(\theta_2)$ with $z_1 \succeq z_2$ and $\theta_2 \succeq \theta_1$ imply $z_2 \in \zeta(\theta_1)$ and $z_1 \in \zeta(\theta_2)$.

Theorem 1 (Nowhere Decreasing Optimizers)**.** *Let $F : Z \times \Theta \mapsto \mathbb{R}$, where Z and Θ are posets, and let $Z' \subseteq Z$. If $\max_{z\in Z'} F(z,\theta)$ exists for all θ and F is single crossing in (z,θ), then $\mathcal{Z}(\theta|Z') \equiv \arg\max_{z\in Z'} F(z,\theta)$ is nowhere decreasing in θ for all Z'. If $\mathcal{Z}(\theta|Z')$ is nowhere decreasing in θ for all $Z' \subseteq Z$, then $F(z,\theta)$ is single crossing.*

($\Rightarrow$): If $\theta_2 \succeq \theta_1$, $z_1 \in \mathcal{Z}(\theta_1)$, $z_2 \in \mathcal{Z}(\theta_2)$, and $z_1 \succeq z_2$, optimality and single crossing give:

$$F(z_1,\theta_1) \geq F(z_2,\theta_1) \quad \Rightarrow \quad F(z_1,\theta_2) \geq F(z_2,\theta_2) \quad \Rightarrow \quad z_1 \in \mathcal{Z}(\theta_2)$$

Now assume $z_2 \notin \mathcal{Z}(\theta_1)$. By optimality and single crossing, we get the contradiction:

$$F(z_1,\theta_1) > F(z_2,\theta_1) \quad \Rightarrow \quad F(z_1,\theta_2) > F(z_2,\theta_2) \quad \Rightarrow \quad z_2 \notin \mathcal{Z}(\theta_2)$$

($\Leftarrow$): Assume $F(z,\theta)$ is not single crossing. Then for some $z_2 \succeq z_1$ and $\theta_2 \succeq \theta_1$: either: (i) $F(z_2,\theta_1) \geq F(z_1,\theta_1)$ and $F(z_2,\theta_2) < F(z_1,\theta_2)$; or, (ii) $F(z_2,\theta_1) > F(z_1,\theta_1)$ and $F(z_2,\theta_2) \leq F(z_1,\theta_2)$. Let $Z' = \{z_1, z_2\}$. In case (i), we have $z_2 \in \mathcal{Z}(\theta_1|Z')$ and $z_1 = \mathcal{Z}(\theta_2|Z')$, which rules out $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_1 \notin \mathcal{Z}(\theta_2|Z')$. In case (ii), we have $z_2 = \mathcal{Z}(\theta_1|Z')$ and $z_1 \in \mathcal{Z}(\theta_2|Z')$, which precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_1 \notin \mathcal{Z}(\theta_1|Z')$. □

B.2 Upcrossing Preservation

Given a Euclidean lattice[21] $Z \subseteq \mathbb{R}^N$ and poset $(\Theta, \succeq)$, the function $\sigma : Z \times \Theta \to \mathbb{R}$ is *proportionately upcrossing* if it obeys inequality (13) $\forall z, z' \in Z$ and $\theta' \succeq \theta$. The measure λ on Z is *non-degenerate for* σ if $\lambda[z : \sigma(z,\theta) = 0] < \lambda(Z)$ for all $\theta \in \Theta$. So λ does not place all mass on zeros of σ.[22]

Theorem 2 (Upcrossing Preservation). *If $\sigma(z,\theta)$ obeys* (13)*, then $\Sigma(\theta) \equiv \int_Z \sigma(z,\theta)d\lambda(z)$ upcrossing in θ if λ is non-degenerate for σ.*

Claim B.1 (Ahlswede and Daykin (1979)). *If $\xi_1, \xi_2, \xi_3, \xi_4 \geq 0$ obey the inequality $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for $z \in Z \subseteq R^N$, then for all positive measures λ:*

$$\int \xi_3(z)d\lambda(z) \int \xi_4(z)d\lambda(z) \geq \int \xi_1(z)d\lambda(z) \int \xi_2(z)d\lambda(z) \tag{16}$$

If $\theta' \succeq \theta$, then (13) is $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for non-negative functions:

$$\xi_1(z) \equiv \sigma^+(z,\theta), \quad \xi_2(z) \equiv \sigma^-(z,\theta'), \quad \xi_3(z) \equiv \sigma^+(z,\theta'), \quad \xi_4(z) \equiv \sigma^-(z,\theta)$$

and thus, by Lemma B.1:

$$\int \sigma^+(z,\theta')d\lambda(z) \int \sigma^-(z,\theta)d\lambda(z) \geq \int \sigma^+(z,\theta)d\lambda(z) \int \sigma^-(z,\theta')d\lambda(z) \tag{17}$$

In addition, when λ is non-degenerate for σ, we have:

$$\Sigma(\theta) \geq 0 \Leftrightarrow \int \sigma^+(z,\theta)d\lambda(z) \geq \int \sigma^-(z,\theta)d\lambda(z) \quad \Rightarrow \quad \int \sigma^+(z,\theta)d\lambda(z) > 0 \tag{18}$$

Combining inequality (17) and (18), we discover that $\Sigma(\theta) \geq 0$ implies $\int \sigma^+(z,\theta')d\lambda(z) \geq \int \sigma^-(z,\theta')d\lambda(z)$, i.e. $\Sigma(\theta') \geq 0$, for all $\theta' \succeq \theta$. □

Toward an easy to check sufficient condition for (13), let $\sigma : \mathbb{R}^N \mapsto \Re$ be *smoothly log-supermodular (LSPM)* if all pairwise derivatives obey $\sigma_{ij}\sigma \geq \sigma_i\sigma_j$.

[21] We prove a stronger than needed result, as it applies to general lattices; we just need it for $\mathbb{R}^2$.

[22] This result is related to Theorem 2 in Quah and Strulovici (2012). They do not assume λ is non-degenerate, but they posit that σ is upcrossing in (z,θ), while we assume that σ is merely weakly upcrossing. This relaxation is critical for us, since σ is only weakly upcrossing in our applications. Do they assume (13)? They do not. Instead, they separately impose a cross sectional and time series signed ratio monotonicity condition. I will typeset exactly what they assume and relate to our condition, and we can decide what to keep.

Lemma B.1. *If $\sigma(z,\theta)$ is upcrossing and smoothly LSPM on $z \in \mathbb{R}^{N-1}$ and $\theta \in \mathbb{R}$, then σ obeys* (13)*.*

Step 1: Ratio Ordering. We claim that if $\sigma : \mathbb{R}^2 \mapsto \mathbb{R}$ is upcrossing with $\sigma_{12}\sigma \geq \sigma_1\sigma_2$, and $\sigma(u_2,v_1) < 0 < \sigma(u_1,v_2)$ for some $(u_1,v_1) < (u_2,v_2)$, then:

$$\frac{\sigma(u_1,v_1)}{\sigma(u_1,v_2)} \leq \frac{\sigma(u_2,v_1)}{\sigma(u_2,v_2)} \tag{19}$$

Indeed, $\sigma_{12}\sigma \geq \sigma_1\sigma_2$ and $v_1 < v_2$ implies:

$$\frac{\sigma_1(u,v_1)}{\sigma(u,v_1)} \leq \frac{\sigma_1(u,v_2)}{\sigma(u,v_2)}$$

but since σ is upcrossing, we know $\sigma(u,v_1) < 0 < \sigma(u,v_2)$ for all $u \in [u_1,u_2]$; and thus cross multiplying, we find:

$$\sigma_1(u,v_1)\sigma(u,v_2) \geq \sigma_1(u,v_2)\sigma(u,v_1) \quad \forall u \in [u_1,u_2]$$

Thus, the ratio $\sigma(u,v_1)/\sigma(u,v_2)$ is non-decreasing in u on $[u_1,u_2]$; and thus (19).

Step 2: σ obeys (13). If z,z' are ordered, then σ upcrossing immediately implies (13). Assume instead that z,z' are unordered and (WLOG) that the RHS of (13) is non-zero for some $\theta' \geq \theta$, i.e. $\sigma(z,\theta') < 0 < \sigma(z',\theta)$. By σ upcrossing, we must also have $\sigma(z \wedge z',\theta) < 0 < \sigma(z \vee z',\theta')$. Altogether given this sign pattern (13) becomes:

$$-\sigma(z \wedge z',\theta)\sigma(z \vee z',\theta') \geq -\sigma(z,\theta')\sigma(z',\theta) \quad \Leftrightarrow \quad \frac{\sigma(z \wedge z',\theta)}{\sigma(z',\theta)} \leq \frac{\sigma(z,\theta')}{\sigma(z \vee z',\theta')}$$

True by sequenced pairwise applications of (19) for $u = z_i$ and $v = z_j$ or $v = \theta$. □

C Omitted Proofs

C.1 Nowhere Decreasing Sorting: Proof of Lemma 3

We apply Lemma 1 to our matching problem. Let F be aggregate output (3), Z be the set of cdfs on $\mathbb{R}^2$ endowed with the PQD order, and $Z' \equiv \mathcal{M}(G,H)$ be the subset of Z with given marginals G and H. Then Lemma 1 yields sorting nowhere decreasing in θ when output (3) is single crossing in (M,θ). The proof of the second sentence in Lemma 3 mirrors the ($\Rightarrow$) step in the proof of Lemma 1 choosing marginals with point mass at two types; so that the constraint set consists of all weighted averages of the two pure matchings $z_2 =$ PAM and $z_1 =$ NAM. □

C.2 Increasing Sorting: Proof of Proposition 2

Recall that we WLOG assume $S(R|\theta)$ is upcrossing in R.

Induction Step 0: Notation and Preliminaries.

- Our general proof is by induction on the number of types.
- The result holds for $n = 2$ types by $S(R|\theta)$ upcrossing in R, since the only pure matchings are the PQD-ranked NAM and PAM, and the PAM payoff is weakly (strictly) higher than the NAM payoff if and only if $S(R|\theta) \geq (>)0$.
- *Induction Assumption:* In any model with $2, 3, \ldots, n$ types, if the sorting premium $S(R|\theta)$ is upcrossing in R and θ and M'' and M' are the unique optimal matchings for $\theta'' \succ \theta'$, then $M'' \succeq_{PQD} M'$.
- We prove the result for $n + 1$ types by showing that the following assumption leads to a contradiction of the Induction Assumption.
- *Contradiction Assumption:* There exists an $n + 1$ type model with sorting premium $S(R|\theta)$ that is upcrossing in θ and R, and $\theta'' \succ \theta'$ with unique pure optimal matchings M' and M'', but not $M'' \succeq_{PQD} M'$.
- Our proof uses the following four Claims.

Claim C.1. *If the sorting premium $S(R|\theta)$ is upcrossing in R and the optimal matching is unique, then no matched NAM pair vector dominates a matched PAM pair.*

Proof: PAM (NAM) is optimal for a pair *iff* $S(R|\theta) \geq (\leq)0$ on the associated rectangle R. Further, we cannot have $S(R|\theta) = 0$ for any optimally matched pair, since this would violate the assumption that the optimal matching is unique. □

- A pure matching can be described in two equivalent ways: specify that woman i is matched to man μ_i for all $i = 1, 2, \ldots, n$, or that man j is matched to woman ω_j for all $j = 1, 2, \ldots, n$.

Claim C.2. *If two pure n type matchings M and $\hat{M}$ are not identical, then $\hat{\mu}_i > \mu_i$ for some i and $\hat{\omega}_j > \omega_j$ for some j.*

Proof: Since pure matchings M and $\hat{M}$ are not identical, there exists a highest type man k, such that $\hat{\omega}_k \neq \omega_k$. Then, since the matchings are pure, the woman matched to man k under matching $\hat{M}$, must be matched to a lower index man under M, i.e. $k = \hat{\mu}_i > \mu_i$ for $i = \hat{\omega}_k$. □

Claim C.3. *Assume the sorting premium is upcrossing in R and θ and Induction Assumption n holds. Removing any woman i and her partner ordered $\mu'_i \geq \mu''_i$ from unique optimal matchings μ' and μ'' at θ' and θ'' with $n+1 \geq 3$ types, results in a matching among the remaining n other types that is PQD higher for θ'' than θ'.*

Proof:

- If an optimal matching is unique with $n+1$ types, then the optimal matching among the remaining n after removing any optimally matched pair is unique.
- If we remove woman i and man j from a market with $n+1$ types, we produce an n type model with sorting premium:

$$S^n_{ij}(R|\theta) \equiv S(R + \Delta_{ij}(R)|\theta) \quad \text{where} \quad \Delta_{ij}(R) = (\mathbb{1}_{I_1 \geq i}, \mathbb{1}_{I_2 \geq j}, \mathbb{1}_{I_3 \geq i}, \mathbb{1}_{I_4 \geq j})$$

- The vector $R + \Delta_{ij}$ increments by one the index of every woman in R whose index is at least i and the index of every man in R whose index is at least j.
- We complete the proof by showing that $S(R|\theta)$ upcrossing in R and θ, implies that $S^n_{iJ(\theta)}(R|\theta)$ is upcrossing in R and θ for all non-increasing functions $J : \mathcal{R} \to \{1, 2, \ldots, N+1\}$. The claimed PQD ordering on the n type matchings then follows by Induction Assumption n.
- By assumption, S is upcrossing in R, while the indicator vector Δ_{ij} is non-decreasing in R. Thus, for any fixed θ, the composite function $S^n_{iJ(\theta)}(R|\theta) \equiv S(I + \Delta_{iJ(\theta)}(R)|\theta)$ is upcrossing in R.
- To see that $S^n_{iJ(\theta)}(R|\theta)$ is upcrossing in θ, consider any $\theta_2 \succ \theta_1$ and assume $S^n_{iJ(\theta_1)}(R|\theta_1) \geq (>)0$; then by definition we have:

$$\begin{aligned} S(R + \Delta_{iJ(\theta_1)}(R)|\theta_1) \geq (>)0 \quad &\Rightarrow \quad S(R + \Delta_{iJ(\theta_2)}(R)|\theta_1) \geq (>)0 \\ &\Rightarrow \quad S(R + \Delta_{iJ(\theta_2)}(R)|\theta_2) \geq (>)0 \\ &\Rightarrow \quad S^n_{iJ(\theta_2)}(R|\theta_2) \geq (>)0 \end{aligned}$$

  The first implication follows from $J(\theta_2) \geq J(\theta_1)$, Δ_{ij} non-increasing in j, and $S(R|\theta)$ upcrossing in R. The second implication follows from $S(R|\theta)$ upcrossing in θ. The last line applies the definition of $S^n_{iJ(\theta_2)}$. □

Claim C.4. *Adding matched couples $(1, m'') \leq (1, m')$ (or $(w'', 1) \leq (w', 1)$) to n type matchings $\mu'' \succeq_{PQD} \mu'$ preserves the PQD order for the resulting $n+1$ type matchings.*

PROOF:

- We consider the $n+1$ type matching induced by adding a given couple to any n type matching.

- For pure matchings μ, let $C_\mu(i,j)$ be the count of matches by women at most i with men at most j. Then $\mu \succeq_{PQD} \mu'$ *iff* $C_\mu \geq C_{\mu'}$.

- Starting with any pure matching μ and adding a matched couple with types between the current $(k-1, \ell-1)$ and (k, ℓ) types, yields a matching with match counts:

$$\mathcal{C}_{k\ell}(\mu)(i,j) \equiv C_\mu\left(i - \mathbb{1}_{i \geq k}, j - \mathbb{1}_{j \geq \ell}\right) + \mathbb{1}_{(i,j) \geq (k,\ell)} \quad \forall i,j \in \{1,2,\ldots,N+1\}$$

- Now, consider two ranked matchings $\hat{\mu} \succeq_{PQD} \mu$. Add a matched couple between current indices $(k-1, \ell-1)$ and (k, ℓ) to μ and between $(\hat{k}-1, \hat{\ell}-1)$ and $(\hat{k}, \hat{\ell})$ to $\hat{\mu}$. The next Lemma asserts that the $N+1$ type matchings will inherit the PQD ranking when $k = \hat{k} = 1$ and $\hat{\ell} \leq \ell$; or alternatively, $\ell = \hat{\ell} = 1$ and $\hat{k} \leq k$.

- In this notation Claim C.4 is: *If two matchings obey $\hat{\mu} \succeq_{PQD} \mu$, then (i) $\mathcal{C}_{1\hat{\ell}}(\hat{\mu}) \geq \mathcal{C}_{1\ell}(\mu)$ for all $\hat{\ell} \leq \ell$; and, (ii) $\mathcal{C}_{\hat{k}1}(\hat{\mu}) \geq \mathcal{C}_{k1}(\mu)$ for all $\hat{k} \leq k$.*

- We will prove implication (i). The logic for case (ii) is symmetric.

- By assumption $\hat{\mu} \succeq_{PQD} \mu$ and thus, $C_{\hat{\mu}} \geq C_\mu$. Thus, for $j < \hat{\ell}$:

$$\mathcal{C}_{1\ell_1}(\hat{\mu})(i,j) - \mathcal{C}_{1k_2}(\mu)(i,j) = C_{\hat{\mu}}(i-1,j) - C_\mu(i-1,j) \geq 0$$

- If instead, $j \geq \ell$, then:

$$\mathcal{C}_{1\ell_1}(\hat{\mu})(i,j) - \mathcal{C}_{1\ell_2}(\mu)(i,j) = C_{\hat{\mu}}(i-1,j-1) - C_\mu(i-1,j-1) \geq 0$$

- Finally, when $\hat{\ell} \leq j < \ell$, the difference $\mathcal{C}_{1\ell_1}(\hat{\mu})(i,j) - \mathcal{C}_{1\ell_2}(\mu)(i,j)$ is:

$$\begin{aligned} & C_{\hat{\mu}}(i-1,j-1) + 1 - C_\mu(i-1,j) \\ = \; & C_{\hat{\mu}}(i-1,j-1) - C_\mu(i-1,j-1) + 1 - [C_\mu(i-1,j) - C_\mu(i-1,j-1)] \\ \geq \; & C_{\hat{\mu}}(i-1,j-1) - C_\mu(i-1,j-1) \geq 0 \end{aligned}$$

where the first inequality follows from $C_\mu(i-1,j) - C_\mu(i-1,j-1) \leq 1$. □

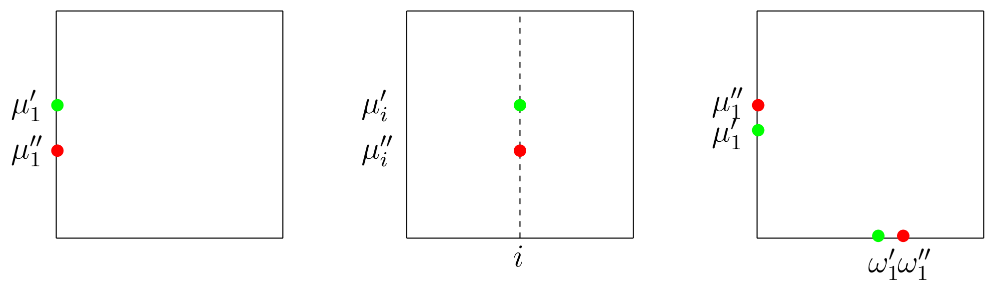


Figure 10: **The Induction Proof Illustrated: Step 1.** We use green dots to denote proposed matched pairs at θ' and red for matched pairs at θ''. The right panel illustrates the conclusion of Step 1 in the proof: to avoid violating the maintained assumption that the $n+1$ type matching is not PQD higher for θ'', the indices of the men matched to the lowest type woman must satisfy $\mu_1'' = \mu_1' + 1$ and the indices of the women matched to the lowest type man must satisfy $\omega_1'' = \omega_1' + 1$.

- We now complete the proof of Proposition 2, using Claims C.1 to C.4 to pin down properties of M' and M'' implied by the Contradiction Assumption (Steps 1-3). We then show that these properties violate the Induction Assumption (Step 4).
- Let μ_i' and μ_i'' be the man matched to woman i for θ' and θ'', respectively. And let ω_j' and ω_j'' be the woman matched to man j for θ' and θ'', respectively.

INDUCTION STEP 1: $\mu_1'' = \mu_1' + 1$ AND $\omega_1'' = \omega_1' + 1$.
We establish the first equality. Symmetric steps prove that $\omega_1'' = \omega_1' + 1$.

STEP 1-A: $\mu_1'' > \mu_1'$. Assume instead that $\mu_1'' \leq \mu_1'$, as in the left panel of Figure 10. Now, remove woman 1 and man μ_1' from the type space for θ' and woman 1 and man μ_1'' for θ''. The matching among remaining n types is PQD higher for θ'' by Induction Assumption n and Claim C.3. Now, returning the optimally matched pairs $(1, \mu_1')$ and $(1, \mu_1'')$, the PQD ranking still holds with $n+1$ types by $\mu_1'' \leq \mu_1'$ and Claim C.4. But this violates our Contradiction Assumption and thus $\mu_1'' > \mu_1'$.

STEP 1-B: $\mu_1'' < \mu_1' + 2$. Assume instead that $\mu_1'' \geq \mu_1' + 2$. By Claim C.2, some woman $i > 1$ exists for whom $\mu_i'' < \mu_i'$ (see middle panel of Figure 10). Remove woman i and her partner μ_i' from the type space at θ', and woman i and her partner μ_i'' from the type space at θ''. Since $\mu_i'' < \mu_i'$, the matching among the remaining n types is PQD higher at θ'' by Claim C.3 and Induction Assumption n. In particular, woman 1 must be matched to a weakly lower index man under θ'' in the n type model. But this is impossible if $\mu_1'' \geq \mu_1' + 2$, since the difference in indices $\mu_1'' - \mu_1'$ falls by at most 1 when we remove man μ_i at θ' and man μ_i'' at θ''.

INDUCTION STEP 2: $\mu'_{\omega_1''} = \mu_1''$. That is, woman ω_1'' and man μ_1'' match together at θ'.

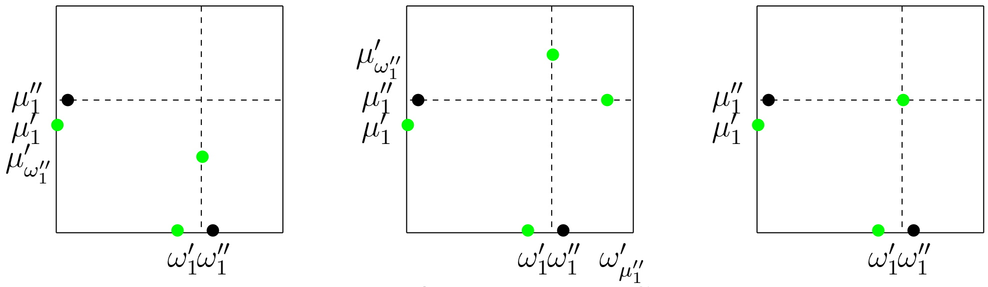


Figure 11: **The Induction Proof Illustrated: Step 2.** This Figure considers the man matched to woman ω_1'' and the woman matched to man μ_1'' under θ'. We use green dots to denote proposed matched pairs at θ' and red for matched pairs at θ''. The right panel illustrates the conclusion of Step 2 in the proof: to avoid violating the maintained assumption that the $n+1$ type matching is not PQD higher for θ'', man μ_1'' and woman ω_1'' must be matched together under θ'.

STEP 2-A: $\omega'_{\mu_1''} \geq \omega_1''$ AND $\mu'_{\omega_1''} \geq \mu_1''$. To prove that $\mu'_{\omega_1''} \geq \mu_1''$, we proceed by contradiction, assuming instead that $\mu'_{\omega_1''} < \mu_1''$. In fact, since $\mu_1' = \mu_1'' - 1$ is already matched, and the matching is pure, we must have $\mu'_{\omega_1''} < \mu_1'$ as illustrated in the left panel in Figure 11. Remove woman ω_1'' and her partner $\mu'_{\omega_1''} \geq 1$ from the type space at θ', and woman ω_1'' and her partner 1 from the type space at θ''. As before, the matching among the remaining types is PQD higher at θ'', by Induction Assumption n and Claim C.3. Since the indices of the man $\mu'_{\omega_1''}$ removed under θ' and the man 1 removed under θ'' are both below μ_1', the ordering of the indices of the male partners for woman 1 is maintained with the n remaining types. In particular, woman 1 is matched to a strictly lower index partner among the n remaining types at θ' But this contradicts the matching among the remaining n types PQD higher at θ''. Thus, $\mu'_{\omega_1''} \geq \mu_1''$ and by symmetric reasoning $\omega'_{\mu_1''} \geq \omega_1''$.

STEP 2-B: CASES RULED OUT BY PURITY. Since we WLOG restrict attention to pure matchings, we need not consider $\omega'_{\mu_1''} > \omega_1''$ and $\mu'_{\omega_1''} = \mu_1''$, since man μ_1'' would then have two partners at θ'. Likewise we need not consider $\omega'_{\mu_1''} = \omega_1''$ and $\mu'_{\omega_1''} > \mu_1''$, since woman ω_1'' would then have two partners at θ'.

STEP 2-C: $\omega'_{\mu_1''} > \omega_1''$ AND $\mu'_{\omega_1''} > \mu_1''$ IS IMPOSSIBLE. This ordering is illustrated in the middle panel of Figure 11. Notice that this implies that the (green) matching for θ' violates the sorting premium up-crossing in the type space. In particular, the green matching includes the PAM pair $(1, \mu_1')$ and $(\omega_1'', \mu'_{\omega_1''})$ and the NAM pair $(\omega_1'', \mu'_{\omega_1''})$ and $(\omega'_{\mu_1''}, \mu_1'')$, violating Claim C.1.

STEP 3: $\mu_1' \geq \mu_i' = \mu_i'' - 1$ FOR $i = 1, \ldots, \omega_1'$ AND $\omega_1' \geq \omega_j' = \omega_j'' - 1$ FOR $j = 1, \ldots, \omega_1'$. We have established the claim for $i = 1$ and $j = 1$, we now prove the claimed ordering $\mu_1' \geq \mu_i' = \mu_i'' - 1$ for $i = 2, \ldots, \omega_1'$. By symmetry, $\omega_1' \geq \omega_j' = \omega_j'' - 1$ for $j = 2, \ldots, \omega_1'$.

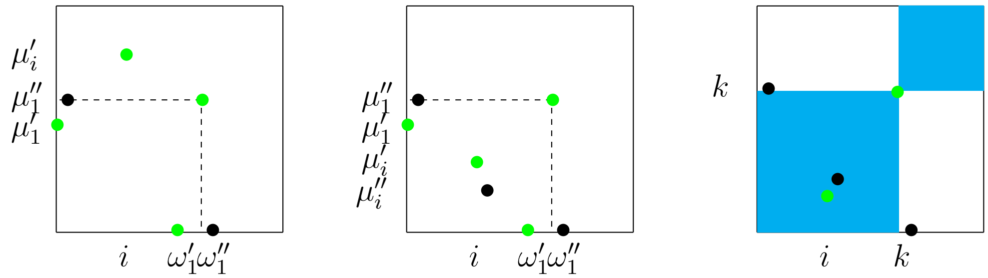


Figure 12: **The Induction Proof Illustrated: Step 3 and 4.** The first two panels incorporate the conclusions of Steps 1 and 2 and consider the ordering of the men μ_i' and μ_i'' matched to woman with index $i \in \{2, \ldots, \omega_1'\}$ as in Step 3. Step 4 establishes that matches only form in the light blue region for θ' and θ'' and that the index of the men matched to type i, obeys $\mu_i'' = \mu_i''$ for all $i = 1, \ldots, k-1$ as shown. Altogether, the matching at θ' PQD dominates the matches at θ'' on the subset of types $1, 2, \ldots, k$.

Step 3-A: $\mu_1' \geq \mu_i'$. On the contrary assume $\mu_1' < \mu_i'$ for some $2 \leq i \leq \omega_1'$. By purity, we cannot have $\mu_i' = \mu_1''$, since μ_1'' is matched to ω_1'' under θ', as established in Step 2. Further, $\mu_1'' = \mu_1' + 1$; and thus, $\mu_1' < \mu_i'$ implies $\mu_1'' < \mu_i'$ as in the left panel of Figure 12. But notice that this implies that the optimal matching for θ' involves a PAM pair $(1, \mu_1')$ and (i, μ_i') below a NAM pair (i, μ_i') and (ω_1'', μ_1''), a violation of S upcrossing in types by Claim C.1.

Step 3-B: $\mu_i' < \mu_i''$. On the contrary assume $\mu_i' \geq \mu_i''$ for some $2 \leq i \leq \omega_1'$ as shown in the middle panel of Figure 12. Since $\mu_i'' \leq \mu_i'$, if we remove woman i and her partner μ_i' from the type space at θ', and woman i and her partner μ_i'' from the type space at θ'', the matching among the remaining types n is PQD higher at θ'', by Claim C.3 and Induction Assumption n. However, woman 1 is matched to a partner with a strictly lower index (among the remaining types) under θ' than under θ'', which contradicts the matching among the remaining n types being PQD higher under θ''.

Step 3-C: $\mu_i' = \mu_i'' - 1$. If we remove woman ω_1'' and her partner man μ_1'' from the type space at θ', and woman ω_1'' and her partner man 1 from the type space at θ'', the matching among the remaining types n is PQD higher at θ'', by Claim C.3 and Induction Assumption n. But, we have have already shown that $\mu_i' < \mu_i''$ for all $i = 1, \ldots, i_1'$. Thus, the matching among the remaining types under θ' has weakly more couples below every pair (i, j) for all $i = 1, \ldots, \omega_1'$, and strictly more couples below index (i, μ_i') if $\mu_i' < \mu_i'' - 1$, contradicting the matching among the remaining n types PQD higher at θ''. Thus, $\mu_i' \geq \mu_i'' - 1$ for all $i = 1, \ldots, \omega_1'$. But then given Step 3-B, we must have $\mu_i' = \mu_i'' - 1$ for all $i = 1, \ldots, \omega_1'$.

Step 4: The Contradiction.

- We have shown that every woman $i \leq \omega_1''$ matches with a man whose index $j \leq \mu_1''$

and every man with an index $j \leq \mu_1''$ matches with a woman whose index $i \leq \omega_1''$. Given purity, we must have $\mu_1'' = \omega_1'' = k$. Altogether, all men and women with indices less than or equal to k match with partners whose indices are also less than or equal to k at *both* θ' and θ''. That is, matches only form in the light blue region in the right panel of Figure 12 under both θ' and θ''.

- Consider the matching among these remaining k types after removing all men and woman with indices above k at *both* θ' and θ''. Since the original matchings were unique, the optimal matching among these lowest k types must also be unique. Thus by the Inductive Assumption n, the matching among the remaining k types must be PQD higher at θ''.

- But, in fact, the matching among the remaining k types at θ' PQD dominates the matching among the remaining k types at θ''. For the matching at θ' has the same number of couples as the matching at θ'' below $(k, k), (k-1, k)$, and $(k, k-1)$, at least as many couples on all lower orthants, and strictly more couples on lower orthants (i, μ_i') for $i < k$ by $\mu_i' = \mu_i'' - 1$ and (ω_j', j) for $j < k$ by $\omega_j' = \omega_j'' - 1$.

C.3 Lemma 5: Uniqueness with Continuum Types

1. We prove the Lemma for G absolutely continuous and $\Delta_x(x|y_1, y_2)$ strictly upcrossing in x. The remaining cases admit symmetric logic.

2. Theorem 5.1 in Ahmad, Kim, and McCann (2011) establishes uniqueness for a C^2 production function, absolutely continuous G and *subtwisted* production; namely, for all y_1, y_2, the *twist difference* $\phi(x, y_2) - \phi(x, y_1)$ has no critical points in x save at most one local max and one local min.

3. Since the subtwist definition allows for both a global max and a global minimum, we can WLOG assume $y_1 < y_2$.

4. Finally, notice that $\Delta_x(x|y_1, y_2) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x implies that the twist difference $\phi(x, y_2) - \phi(x, y_1)$ can have at most one critical point, necessarily a global minimum. □

C.4 Proof of Proposition 3

PROOF OUTLINE: We construct a sequence of finite type models, establish that monotone sorting obtains along this sequence, and then show that the sequence of optimal matchings converges to a continuum limit with monotone sorting. Throughout the proof we WLOG assume one-crossing means upcrossing.

Step 1: $S(R|\theta)$ is Strictly Upcrossing in R. In the text we proved $S(x_1, y_1, x_2, y_2)) = \int_{x_1}^{x_2} \Delta_x(x|y_1, y_2)dx$ is upcrossing in (x_1, x_2). Given $\Delta_x(x|y_1, y_2)$ strictly upcrossing in x, $S(x_1', y_1, x_2', y_2) = 0$ implies $\Delta_x(x_1'|y_1, y_2) < 0 < \Delta_x(x_2'|y_1, y_2)$; and thus, the derivatives obey $S_{x_1}(x_1', y_1, x_2', y_2) = -\Delta_x(x_1'|y_1, y_2) > 0$ and $S_{x_2}(x_1', y_1, x_2', y_2) = \Delta_x(x_2'|y_1, y_2) > 0$. Thus, $S(x_1'', y_1, x_2'', y_2) > 0$ for all $(x_1'', x_2'') > (x_1', x_2')$. Symmetric logic establishes S also strictly upcrossing in (y_1, y_2); and thus, S strictly upcrossing in R.

Step 2: A Sequence of Uniquely Optimal Finite Type Matchings.

1. Define the *sequence of n type models* associated with the continuum model, such that types $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ equalize quantile increments: $G(x_1) = 1/(2n)$ and $G(x_i) = G(x_{i-1}) + 1/n$ and $H(y_1) = 1/(2n)$ and $H(y_j) = H(y_{j-1}) + 1/n$; and the output function is $f_{ijn}(\theta) = \phi(x_i, y_j|\theta)$.

2. Fix $\theta'' \succ \theta'$ and let M_n' and M_n'' be pure matchings optimal for $f_{ijn}(\theta')$ and $f_{ijn}(\theta'')$.

3. Let $p_{ijn}(\theta)$ be the indicator function on (i, j) matched at $M_n(\theta)$ for $\theta \in \{\theta', \theta''\}$.

4. Note that for *any* $\varepsilon > 0$, M_n'' and M_n' are uniquely optimal for payoffs:

$$f_{ijn}^{\varepsilon}(\theta) \equiv f_{ij}(\theta) + \varepsilon p_{ijn}(\theta) \qquad \theta \in \{\theta', \theta''\}$$

5. Let $s_{ijn}^{\varepsilon}(\theta)$ and $S_n^{\varepsilon}(R|\theta)$ be the synergy and sorting premium associated with $f_{ijn}^{\varepsilon}(\theta)$.

Step 3: $M_n'' \succeq_{PQD} M_n'$ for all n.

1. Since the sorting premium $S_n^{\varepsilon}(R|\theta)$ is continuous in ε; and there is a finite grid of rectangles $R = (i_1, j_1, i_2, j_2)$ for all n, there exists $\varepsilon_n^* > 0$ such that, for all $\varepsilon < \varepsilon_n^*$, we cannot have any strict sign changes: $S_n^0(R|\theta) < 0 \Rightarrow S_n^{\varepsilon}(R|\theta) < 0$ and $S_n^0(R|\theta) > 0 \Rightarrow S_n^{\varepsilon}(R|\theta) > 0$; or equivalently:

$$S_n^{\varepsilon}(R|\theta) \geq 0 \;\Rightarrow\; S_n^0(R|\theta) \geq 0 \quad \text{and} \quad S_n^{\varepsilon}(R|\theta) \leq 0 \;\Rightarrow\; S_n^0(R|\theta) \leq 0 \qquad (20)$$

2. First, we claim that $S_n^{\varepsilon}(R|\theta)$ is strictly upcrossing in R for all $\varepsilon < \varepsilon_n^*$. Assume not: then there exists $R'' > R'$ such that $S_n^{\varepsilon}(R''|\theta) \leq 0 \leq S_n^{\varepsilon}(R'|\theta)$. But then by (20), we must also have: $S_n^0(R''|\theta) \leq 0 \leq S_n^0(R'|\theta)$, contradicting $S_n^0(R|\theta)$ strictly upcrossing in R (Step 1).

3. We also claim that $S_n^\varepsilon(R|\theta)$ is strictly upcrossing in $\theta \in \{\theta', \theta'')$ for all $\varepsilon < \varepsilon_n^*$. Assume not: then there exists R such that $S_n^\varepsilon(R|\theta'') \leq 0 \leq S_n^\varepsilon(R|\theta)$. But then by (20), we must also have: $S_n^0(R|\theta'') \leq 0 \leq S_n^0(R|\theta')$, contradicting $S_n^0(R|\theta)$ strictly upcrossing in θ (assumed).

4. Altogether, for all $\varepsilon \in (0, \varepsilon_n^*)$, the sorting premium $S_n^\varepsilon(R|\theta)$ is upcrossing in R and $\theta \in \{\theta', \theta''\}$ and M_n' and M_n'' are uniquely optimal (Step 2). Altogether, by Proposition 2, we have $M_n'' \succeq_{PQD} M_n'$ for all n.

Step 4: Convergent Sequences. As usual, we say the sequence of cdfs $\{\pi_k\}$ weakly converges to π if $\int u d\pi_k$ converges to $\int u d\pi$ for all bounded continuous functions u. We make use of the following special case of Theorem 5.20 in Villani (2008):

Claim C.5. *Let $\{\phi_k\}$ be a sequence of continuous and uniformly bounded production function converging uniformly to ϕ. Let $\{G_k\}$ and $\{H_k\}$ be a sequence of cdfs and let M_k be an optimal matching given ϕ, G_k, and H_k. If G_k and H_k weakly converge to G and H, then there exists a subsequence of $\{M_k\}$ that weakly converges to some matching M^*, that is optimal for ϕ, G, and H.*

Step 5: PQD Ordering in the Limit.

1. Consider the sequence of n type models defined in Step 2 with $\varepsilon_n = \varepsilon_n^*/n$, defined in Step 3, along with the associated uniquely optimal matchings M_n' and M_n''.

2. By construction the n type discrete distributions weakly converge to G and H, while the discrete production functions $\{f_{ijn}^{\varepsilon_n}(\theta'), f_{ijn}^{\varepsilon_n}(\theta'')\}$ converge uniformly to $\phi(x, y|\theta')$ and $\phi(x, y|\theta'')$. Thus, by Claim C.5, there exists a subsequence of $\{(M_n', M_n'')\}$ that converges to $(M^*(\theta''), M^*(\theta'))$ optimal in the continuum model. Since the premise of Proposition 3 implies the premise of Lemma 5, these limits are the unique optimal matchings in the continuum model.

3. Finally, we must have $M^*(\theta'') \succeq_{PQD} M^*(\theta')$. For, if not, then there must be some (x, y) with $M^*(x, y)(\theta'') < M^*(x, y)(\theta')$. But then along the convergent subsequence of matchings, there exists $n < \infty$ such that $M_n''(\hat{x}, \hat{y}) < M_n'(\hat{x}, \hat{y})$ for some $(\hat{x}, \hat{y})$ close to (x, y), violating $M_n'' \succeq_{PQD} M_n'$ for all n (Step 3). □

C.5 Proof of Lemma 6

Throughout we WLOG assume that one-crossing means upcrossing.

Step 0: Lebesque Measures are Non-Degenerate for Synergy. We will apply Theorem 2 for synergy integrated on Lebesque measures on non-empty rectangles in $\mathbb{R}^2$. Trivially, such measures are non-degenerate for generic finite type synergy (case (a)) and any strictly upcrossing synergy function (case (b)).

Step 1: $S(R|\theta)$ is upcrossing in R and θ (Part (a)). Rewrite (12) as:

$$S(R|\theta) = \int_0^1 \int_0^1 \phi_{12}(x, y|\theta) \mathbb{1}_{(x,y)\in R} dx dy$$

Easily, if $(x, y) \in R$ and $(x', y') \in R'$, then $(x, y) \wedge (x', y') \in R \wedge R'$ and $(x, y) \vee (x', y') \in R \vee R'$; and thus, the indicator function $\mathbb{1}_{z\in R}$ is log-spm in (R, x, y), Consequently, the product $\sigma(z, t) = \phi_{12}(z|\theta)\mathbb{1}_{z\in R}$ obeys (13) for both $t = \theta$ and $t = R$ *iff* synergy does.

Altogether, if synergy is proportionately upcrossing, then the sorting premium $S(R|\theta)$ given by (12) is upcrossing in R and θ by Lemma 2.

Step 2: Strict Upcrossing Conditions in the Continuum Case (Part (b)).

Claim C.6. *Let σ be a strictly upcrossing function of (z, θ), Then $\int_{z_1}^{z_2} \sigma(z, \theta)dz$ is strictly upcrossing in θ if $\int \sigma(z, \theta)\mathbb{1}_{z\in\mathcal{I}_1\cup\mathcal{I}_2} dz$ is upcrossing in θ for all intervals $\mathcal{I}_1, \mathcal{I}_2$.*

Proof: Toward a contradiction, assume $\int_{z_1}^{z_2} \sigma(z, \theta)dz$ is upcrossing, but not strictly upcrossing, in θ. Then, there exists $\theta'' \succ \theta'$ with :

$$\int_{z_1}^{z_2} \sigma(z, \theta')dz = \int_{z_1}^{z_2} \sigma(z, \theta'')dz = 0 \tag{21}$$

Further, by $\sigma(z, \theta)$ strictly upcrossing in z and θ, we have $\sigma(z, \theta) \lesseqqgtr 0$ as $z \lesseqqgtr z^*(\theta)$ for $\theta \in \{\theta', \theta''\}$ with $z_1 < z^*(\theta'') < z^*(\theta') < z_2$. Thus by (21) and $\int \sigma(z, \theta)\mathbb{1}_{z\in\mathcal{I}_1\cup\mathcal{I}_2} dz$ upcrossing in θ:

$$\begin{aligned} 0 &= \textstyle\int_{z_1}^{z_2} \sigma(z, \theta')dz < \int \sigma(z, \theta')\mathbb{1}_{z\in[z_1,z^*(\theta'')]\cup[z^*(\theta'),z_2]} dz \\ \Rightarrow\ 0 &< \textstyle\int \sigma(z, \theta'')\mathbb{1}_{z\in[z_1,z^*(\theta'')]\cup[z^*(\theta'),z_2]} dz < \int_{z_1}^{z_2} \sigma(z, \theta'')dz \end{aligned}$$

Altogether $\int_{z_1}^{z_2} \sigma(z, \theta'')dz > 0$, contradicting (21). □

Step 2-A: MPIX and MPIY. Since syynergy obeys (13) and the indicator function $\mathbb{1}_{y\in[y_1,y_2]}$ is log-spm in (y, y_1, y_2), the product $\sigma(z, t) = \phi_{12}(x, z|\theta)\mathbb{1}_{z\in[y_1,y_2]\cup[\hat{y}_1,\hat{y}_2]}$ obeys (13) for $t = (x, y_1, y_2, \hat{y}_1, \hat{y}_2, \theta)$, as a consequence the integral $\int \phi_{12}(x, y|\theta)\mathbb{1}_{y\in\mathcal{I}_1\cup\mathcal{I}_2} dy$ is upcrossing in x and θ for all intervals $\mathcal{I}_1, \mathcal{I}_2$ by Theorem 2, and thus, since synergy $\phi_{12}(x, y|\theta)$ is strictly upcrossing in (x, y, θ), the integral $\Delta_x(x|y_1, y_2, \theta) = \int_{y_1}^{y_2} \phi_{12}(x, y|\theta)dy$ is strictly upcrossing in x and θ by Claim C.6. Symmetric logic establishes MPIY.

Step 2-B: $S(R|\theta)$ is strictly upcrossing in θ. Consider the integral:

$$\int \Delta_x(x, y_2, y_1, \theta)\mathbb{1}_{x\in\mathcal{I}_1\cup\mathcal{I}_2}dx = \int \left[\int \phi(x, y|\theta)\mathbb{1}_{y\in[y_1,y_2]}dy\right]\mathbb{1}_{x\in\mathcal{I}_1\cup\mathcal{I}_2}dx$$

Again, since the indicator function is log-spm and synergy obeys (13), this integral is upcrossing in θ for all $\mathcal{I}_1, \mathcal{I}_2$ by Theorem 2. But then since $\Delta_x(x, y_2, y_1, \theta)$ is strictly upcrossing in x and θ (Step 2-A), Claim C.6 yields $S(R|\theta) \equiv \int_{x_1}^{x_2} \Delta_x(x, y_2, y_1, \theta)dx$ strictly upcrossing in θ. □

D Omitted Algebra for Economic Applications

D.1 Kremer-Maskin Example

Corollary 1. *In the smooth KM model, sorting is nowhere-decreasing in α and $1/\varrho$.*

Step 1: PAM obtains *iff* $\varrho < (1 - 2\alpha)^{-1}$. Recall that PAM is optimal in the unisex *iff* the symmetric sorting surplus is globally positive. In the smooth KM model, the sign of the symmetric sorting surplus is constant along any ray $y = kx$ and proportional to:

$$s(k) \equiv 2^{\frac{1-2\alpha}{\varrho}}(1 + k) - 2k^\alpha(1 + k^\varrho)^{\frac{1-2\alpha}{\varrho}} \tag{22}$$

Clearly, $s(1) = 0$, while $s'(1) = 0$ and $s''(1) \propto (1 + \rho(2\alpha - 1))$, implying that $s(k)$ is negative close to $k = 1$ precisely when $\varrho > (1 - 2\alpha)^{-1}$. In other words, the sorting premium is negative in a cone around the diagonal and PAM cannot obtain. Conversely, when $\varrho < (1-2\alpha)^{-1}$, $s(k)$ is positive near $k = 1$, but then it is positive for all $k \in [0, 1]$, since s obeys: (i) upcrossing condition $s(k_2) \geq 0$ implies $s(k_1) \geq 0$ for $0 < k_1 < k_2 < 1$; and (ii) $s(0) > 0$. Finally, sorting surplus is symmetric about the 45 degree line, so $s(k) \geq 0$ on $[0, 1]$ implies that the sorting premium is globally positive and PAM is optimal.

Step 2: A One Dimensional Integral. Using symmetry, $\phi_{xy}(x, y) = \phi_{xy}(y, x)$ and change of variable $y = kx$ we rewrite weighted synergy $\varphi(\lambda|\theta, \varrho)$:

$$\begin{aligned}\int\int \phi_{xy}\lambda(dx, dy) &= \int_0^1\int_0^1 \phi_{xy}(x, y)\lambda(x, y)dydx = 2\int\int^x \phi_{xy}(x, y)\lambda(x, y)dydx \\ &= 2\int_0^1\int_0^1 x\phi_{xy}(x, kx)\lambda(x, kx)dkdx = \int_0^1\int_0^1 \gamma(\theta, \varrho, k)\lambda(x, kx)dxdk\end{aligned}$$

where $\gamma(\theta, \varrho, k)$ does not depend on x;[23] and thus: What the heck does this have to do

[23] Algebra to be typeset.

with synergy?

$$\varphi = \int_0^1 \gamma(\theta, \varrho, k) \left[\int_0^1 \lambda(x, kx)dx\right] dk \equiv \int_0^1 \gamma(\theta, \varrho, k)\Delta(k)dk$$

where $\Delta(k) \equiv \int_0^1 \lambda(x, kx)dx \geq 0$.

Step 3: γ is downcrossing in k. Algebra to be typeset.

Step 4: γ is upcrossing in θ. Firstly, by Step 2, the only possible sign disagreement for $k_1 < k_2$ is $\gamma(k_2, \theta, \varrho) < 0 < \gamma(k_1, \theta, \varrho)$. Secondly, $[\log(\gamma)]_{\theta k} \geq 0$ when $\gamma > 0$ and $[\log(-\gamma)]_{\theta k} \leq 0$ when $\gamma < 0$; and thus, the ratio $\gamma(k_2, \theta, \varrho)/\gamma(k_1, \theta, \varrho)$ is non-decreasing in θ. Replace the following with our condition using the Lemma below.Altogether, $\gamma(k_2, \theta, \varrho)$ and $\gamma(k_1, \theta, \varrho)$ obey signed ratio monotonicity, yielding $\int_0^1 \gamma(\theta, \varrho, k)\Delta(k)dk$ upcrossing in θ for all $\Delta(k) \geq 0$ by Theorem 1 in Quah and Strulovici (2012). And thus, weighted synergy $\varphi(\theta, \lambda)$ is upcrossing in θ for all densities λ.

Step 4: φ is downcrossing in ρ. Outline: We apply second change of variable $z = k^\rho$ to $\int_0^1 \gamma(\theta, \varrho, k)\Delta(k)dk$, to express φ as $\int_0^1 g(\theta, \varrho, z)\Delta(z)dz$. We show that g is upcrossing in $1 - z$ and Log-SPM in $(z, -\varrho)$. Altogether, $\varphi(\varrho, \lambda) = \int_0^1 g(\theta, \varrho, z)\Delta(z)dz$ is upcrossing in $-\rho$ by Quah and Strulovici (2012). □

D.2 Partnership Model

We've shown that sorting falls in technology κ if either: (i) $\rho \geq 1/2$ or (ii) or $\rho \in [0, 1 - \eta]$. Indeed, when $\rho > 0$, the function $g(x, y|\kappa) \equiv (\rho - \eta)\kappa^\rho + (1 - \eta)\ell(x, y)^\rho$ is strictly increasing in (x, y) and strictly decreasing in κ; and thus synergy is strictly upcrossing in (x, y) and strictly downcrossing in κ. When $\rho \geq 1/2$ (case (i)), ς is log-spm in the triple (x, y, κ). As the product of a log-SPM function of (x, y, κ) and a strictly monotone function of (x, y, κ), synergy is proportionately upcrossing. When $\rho \in (0, 1 - \eta]$ (case (ii)), synergy is decreasing in κ and the product of a function that is log-spm in (x, y) and a function that is strictly rising in (x, y). Altogether, we have shown that in both case (i) and case (ii) synergy is proportionately upcrossing and strictly upcrossing in $(x, y, -\kappa)$.

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