Lottery Equilibrium

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Columbia Market Design Conference

April 13, 2018

Introduction

- Indivisibilities and non-convex preferences often present problems:
  - general equilibrium theory
  - market design

- **Goal:** develop a unified and simple approach to these problems

Introduction

General equilibrium theory

- With indivisibilities and/or non-convex preferences:
  - competitive equilibria may fail to exist
  - competitive equilibria may be inefficient (in a sense)
- Simple solution:
  - allow traders to engage in (binary) lotteries
  - "lottery equilibrium"

Related literature

- **Lottery equilibrium in special cases:**
  - Rogerson (1988)
  - Hylland and Zeckhauser (1979); Budish, Che, Kojima and Milgrom (2013); Akbarpour and Nikzad (2017)
- **Competitive equilibrium from equal incomes:**
  - Varian (1974)
  - Budish (2011); Budish and Kessler (2016); Budish, Cachon, Kessler and Othman (2017)
- **Competitive equilibrium in continuum economies:**
  - Mas-Colell (1977)

Outline

- Example
- Model (Continuum Economy)
- Results
  - Existence
  - First Welfare Theorem
  - Second Welfare Theorem
- Next Steps
  - Finite Economy
  - Market Design Applications

Example

Environment

- **Consumption set:** $\Omega = \mathbb{Z}_{\geq 0} \times [0, \infty) \times [0, \infty)$
  - one indivisible good "houses"
  - one divisible good "corn"
  - one divisible "artificial currency"
- **Binary lotteries:** $\Delta(\Omega)$
- **Agents:** $t \in T = [0, 1]$
  - utility function $u_t(a_t) = 3(1 + t)\mathbb{1}(a_t^1 \geq 1) + a_t^2$
  - endowment $\omega_t \in \Omega$

$$\underbrace{\int \omega_t}_{\substack{\text{"inside"} \\ \text{endowment}}} + \underbrace{\int \psi_t}_{\substack{\text{"outside"} \\ \text{endowment}}} = \left(\frac{1}{2}, 1, 1\right)$$

Example

Competitive equilibrium vs. lottery equilibrium

Endowments (for now):

- no outside endowments: $\int \psi_t = (0, 0, 0)$
- inside endowments: $\omega_t \in \{(1, 0, 1), (0, 2, 1)\}$

Competitive equilibrium allocation:

- agents consume their endowments
- Pareto dominated (by a lottery allocation)

Example

Competitive equilibrium vs. lottery equilibrium

Endowments (for now):

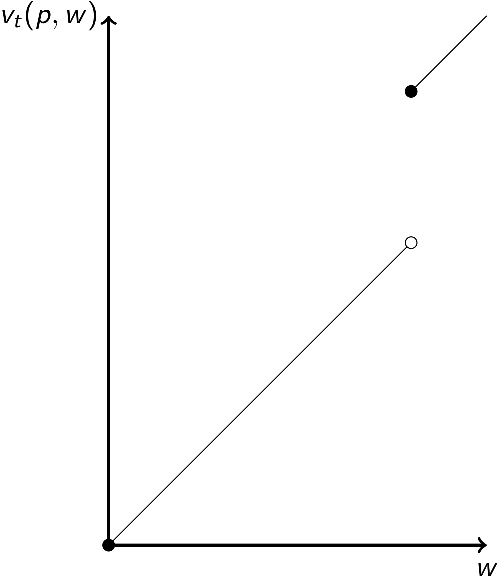
- no outside endowments: $\int \psi_t = (0,0,0)$
- inside endowments: $\omega_t \in \{(1,0,1),(0,2,1)\}$

Lottery equilibrium allocation:

- $t \leq \frac{1}{3} : a_t = \begin{cases} (0,4,1) & \text{if } \omega_t = (1,0,1) \\ (0,2,1) & \text{if } \omega_t = (0,2,1) \end{cases}$
- $t > \frac{1}{3} : a_t = \begin{cases} (1,0,1) & \text{if } \omega_t = (1,0,1) \\ \frac{1}{2} \cdot (1,0,1) + 0 \cdot (0,0,1) & \text{if } \omega_t = (0,2,1) \end{cases}$
- Pareto efficient

Binary lotteries suffice

$t = 1,\ p = (\frac{4}{5}, \frac{1}{5}, 0)$



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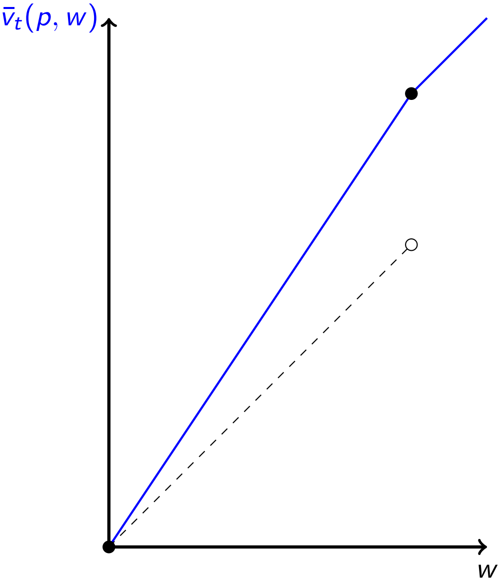

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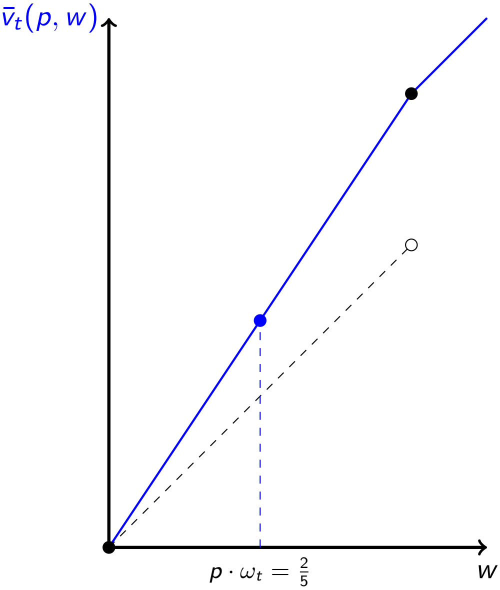

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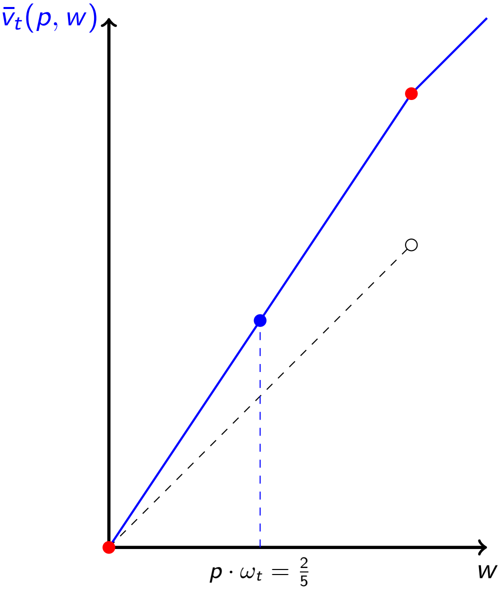

Example

Envy-freeness

An (ex ante) envy-free allocation:

- $\forall t : a_t = \frac{1}{2} \cdot (1, 1, 0) + \frac{1}{2} \cdot (0, 1, 0)$

Example

Envy-freeness

An (ex ante) envy-free allocation:

- $\forall t : a_t = \frac{1}{2} \cdot (1, 1, 0) + \frac{1}{2} \cdot (0, 1, 0)$

The efficient and envy-free allocation:

- $t \leq \frac{1}{3} : a_t^* = (0, 3, 0)$
- $t > \frac{1}{3} : a_t^* = \frac{3}{4} \cdot (1, 0, 0) + \frac{1}{4} \cdot (0, 0, 0)$

Example

Second Welfare Theorem

Failure of 2WT:

- suppose outside endowments are $\int \psi_t = (0, 0, 1)$ and inside endowments satisfy $\int \omega_t = (\frac{1}{2}, 1, 0)$
- for all inside endowments $\omega : T \to \Omega$ and all price vectors p, (p, a^*) is not a lottery equilibrium

Success of 2WT:

- suppose outside endowments are $\int \psi_t = (\frac{1}{2}, 1, 0)$ and inside endowments are $\omega : t \mapsto (0, 0, 1)$
- $\big((\frac{1}{2}, \frac{1}{8}, \frac{3}{8}), a^*\big)$ is a lottery equilibrium

Example

Second Welfare Theorem

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Success of 2WT:

- suppose outside endowments are $\int \psi_t = (\frac{1}{2}, 1, 0)$
  and inside endowments are $\omega : t \mapsto (0, 0, 1)$
- $\left((\frac{1}{2}, \frac{1}{8}, \frac{3}{8}), a^*\right)$ is a lottery equilibrium
- (in contrast, competitive equilibrium fails to exist)

Continuum model

Environment

- **Consumption set:** $\Omega = \mathbb{Z}^m_{\geq 0} \times [0, \infty)^n \times [0, \infty)$
  - m indivisible goods
  - n divisible goods
  - one divisible "artificial currency"
- **Binary lotteries:** $\Delta(\Omega)$
- **Agents:** $t \in T = [0, 1]$
- **Economy:** $e : T \to \mathcal{U} \times \Omega \times \Omega$
  - u_t: agent's utility function (continuous, weakly increasing, constant in last component)
  - ω_t: agent's inside endowment
  - $\int \psi_t$: aggregate outside endowment

Continuum model

Lottery allocations

Lottery allocation: $a : T \to \Delta(\Omega)$ such that

$$\int \mathbb{E}[a_t] \leq \int \omega_t + \int \psi_t$$

- with equality in the first $m + n$ components

Continuum model

Lottery allocations

Lottery allocation: $a : T \to \Delta(\Omega)$ such that

$$\int \mathbb{E}[a_t] \leq \int \omega_t + \int \psi_t$$

- with equality in the first $m + n$ components

(Ex ante) Pareto efficiency: there is no other lottery allocation a' such that

- $u_t(a'_t) \geq u_t(a_t)$ for all $t \in T$
- $u_t(a'_t) > u_t(a_t)$ for all $t \in T' \subset T$, $\lambda(T') > 0$

(Ex ante) envy-freeness: $u_t(a_t) \geq u_t(a_s)$ for all $(s, t) \in T \times T$

Continuum model

Lottery equilibrium

Lottery equilibrium: (p, a) where $p \in \Delta$ and a is a lottery allocation such that for all $t \in T$:

$$
\begin{aligned}
a_t &\in B_t(p) := \{a \in \Delta(\Omega) : p \cdot \mathbb{E}[a] \leq p \cdot \omega_t\} \\
a_t &\in C_t(p) := \underset{a \in \Delta(\Omega) \cap B_t(p)}{\arg\max} \; u_t(a) \\
a_t &\in D_t(p) := \underset{a \in \Delta(\Omega) \cap C_t(p)}{\arg\min} \; p \cdot \mathbb{E}[a]
\end{aligned}
$$

Existence

Theorem
If an economy e satisfies either Condition (A) or Condition (B), then there exists a lottery equilibrium (p, a) for e.

Condition (A)
- each u_t is strictly monotonic in the first $m + n$ components
- each u_t is bounded above by a strictly concave function

Condition (B)
- each u_t is satiated by some $\bar{a} \in \Omega$
- each $\omega_t^{m+n+1} > 0$

First Welfare Theorem

Theorem

If (p, a) is a lottery equilibrium for e, then a is Pareto efficient.

Second Welfare Theorem

Theorem

If

- $e = (u, \omega, \psi)$ *is an economy satisfying Condition (A)*
- a^* *is a Pareto efficient lottery allocation with* $a_t^{*,m+n+1} = 0$ *for all* t

Then there exists an economy $\hat{e} = (\hat{u}, \hat{\omega}, \hat{\psi})$ *such that*

- $\hat{u} = u$ *and* $\int \hat{\omega}_t + \int \hat{\psi}_t = \int \omega_t + \int \psi_t$
- (p, a^*) *is a lottery equilibrium for* $\hat{e}$ *for some prices* $p \in \Delta$

Lottery equilibrium from equal incomes (LEEI)

Lemma

If (p, a) is a lottery equilibrium for an economy e in which $\omega : T \to \Omega$ is a constant mapping, then a is envy-free.

Lottery equilibrium from equal incomes (LEEI)

Lemma

If (p, a) is a lottery equilibrium for an economy e in which $\omega : T \to \Omega$ is a constant mapping, then a is envy-free.

Theorem

If an economy e satisfies either Condition (A) or Condition (B), then there exists a lottery allocation for e that is both envy-free and Pareto efficient.

Proof.

- Reallocate the artificial currency equally across agents
- Reallocate all other goods to the outside endowment
- Compute a lottery equilibrium $\square$

Combinatorial allocation

A-LEEI

Setting: a set of goods (e.g. courses) to allocate among a set of agents (e.g. students) who demand bundles (e.g. schedules)

A-LEEI mechanism:

1. Ask agents to report their utility functions
2. Consider a continuum replication of the setting
3. Compute a lottery equilibrium from equal incomes, which determines a lottery for each original agent
4. Resolve lotteries and assign agents their bundles

Combinatorial allocation
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- "Approximate" because there will be some market clearing error conjectured convergence rates

Previous approaches to combinatorial allocation

Other versions of (A)-LEEI

Paper	Constraints	Utilities	Clearing error
Hylland and Zeckhauser (1979)	capacity	unit demand	none
Budish, Che, Kojima and Milgrom (2013)	bihierarchy	additive	none
Akbarpour and Nikzad (2017)	general	additive	small

Previous approaches to combinatorial allocation

A-CEEI

- **A-CEEI:** approximate *competitive* equilibrium from equal incomes
    - Budish (2011)
    - Budish and Kessler (2016); Budish, Cachon, Kessler and Othman (2017)

- "Approximate" because
    - there will be some market clearing error
    - incomes cannot be perfectly equal

Social lotteries

- In economies with non-convexities, lotteries concavify indirect utility functions
  - efficiency gains (Friedman and Savage, 1948)
  - strengthens the benefits of social insurance
- Suggests that governments should offer menus of actuarially fair "social lotteries"
  - binary lotteries would suffice
  - certain safeguards might be appropriate

Summary

- With indivisibilities and/or non-convex preferences, it can be costly to prohibit trades of probability shares of bundles
  - existence
  - first welfare theorem
  - second welfare theorem

Next steps

Investigate properties of A-LEEI:

- Bound the rate at which clearing error diminishes in finite economies as the market grows
- Empirical comparison to A-CEEI (Budish and Kessler, 2016)

Explore other applications:

- Dynamic allocation
- Two-sided matching

Back-Up Slides

Convergence rates conjectures

back

- Apply the A-LEEI mechanism to the K-fold replication of a fixed finite economy
- Clearing error (as a fraction of the total supply) should be
  - $O\left(\frac{1}{\sqrt{K}}\right)$ for each good, except with probability that is $O(e^{-K})$
  - $O\left(\frac{1}{\sqrt{K}}\right)$ *for all goods uniformly*, except with probability that is $O(e^{-\frac{K}{m+n}})$

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