Minimizing Justified Envy in School Choice: The Design of New Orleans' One App

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Top Trading Cycles in Prioritized Matching: An Irrelevance of Priorities in Large Markets

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  1. **Pareto-efficiency:** maximizing agents' welfare
  2. **Stability:** respecting agents' priorities/eliminating justified envy.

Our questions

Fix one of the criterion, fulfill as much as possible the other.

- If the policy maker prefers the elimination of justified envy. DA is the "natural" choice (maximizes efficiency)
- On the other hand, if the policy maker prefers efficiency, alternative abound: TTC, SD,...
    - Many Pareto-efficient mechanisms, how do they compare in terms of justified envy?

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- If the policy maker prefers the elimination of justified envy. DA is the "natural" choice (maximizes efficiency)
- On the other hand, if the policy maker prefers efficiency, alternative abound: TTC, SD,...
  - Many Pareto-efficient mechanisms, how do they compare in terms of justified envy?
- Raises an even more basic question. TTC has been recommended (and adopted). TTC "uses" priorities but in what sense does it improve on mechanisms ignoring priorities? (Like RSD)

Preview

- In one-to-one environments:
  - there is no Pareto efficient and strategy proof mechanism that has less justified envy than TTC.
- In the many-to-one environment:
  - TTC (and existing variants) are all envy-dominated by alternative Pareto efficient and strategy proof mechanisms.
  - In expectation, TTC has less justified envy than RSD
- In Large Markets
  - Strong equivalence result between TTC and RSD in large economies.
  - Large nb. schools: TTC is not significantly different from RSD in terms of justified envy

Model

A school choice problem consists of:

1. a set of students $I = \{i_1, ..., i_n\}$,
2. a set of schools $S = \{s_1, ..., s_m\}$,
3. a capacity vector $q = (q_{s_1}, ..., q_{s_m})$,
4. a list of strict student preferences $P = (P_{i_1}, ..., P_{i_n})$, and
5. a list of strict school priorities $\succ = (\succ_{s_1}, ..., \succ_{s_m})$.

- $\mu(i) \in S \cup \{i\}$ is student i's match
- Pareto efficiency, justified envy (blocking), matching mechanism, strategy-proofness

Top Trading Cycles

As long as there are schools with available seats and students who are not yet assigned:

1. each unassigned student points to her most preferred available school in her choice list,
2. each available school points to the highest ranked unassigned student in its rank list,
3. a cycle of schools and students pointing to one another exits:

$$i_1 \rightarrow s_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n \rightarrow s_n \rightarrow i_1$$

4. each student in a cycle is assigned a seat at the school she points to
5. a school becomes unavailable of if all of its seats are assigned

Comparing Mechanisms

Definition

A mechanism φ_1 has **less justified envy** than φ_2 at priority profile $\succ$, if for any preference profile P and student-school pair (i, s), if pair (i, s) blocks $\varphi_1(P, \succ)$, then pair (i, s) blocks $\varphi_2(P, \succ)$.

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Definition

A mechanism φ_1 has **strictly less justified envy** than φ_2 if φ_1 has less envy than φ_2, but φ_2 does not have less envy than φ_1.

Minimal Envy

Since the concept of less justified envy defines a preorder, our last definition describes the minimal element of that order.

Definition

Given a class of mechanisms $\mathcal{C}$, φ is **justified envy minimal** in $\mathcal{C}$ if there is no other mechanism ψ in $\mathcal{C}$ that has strictly less envy than φ.

Main Result

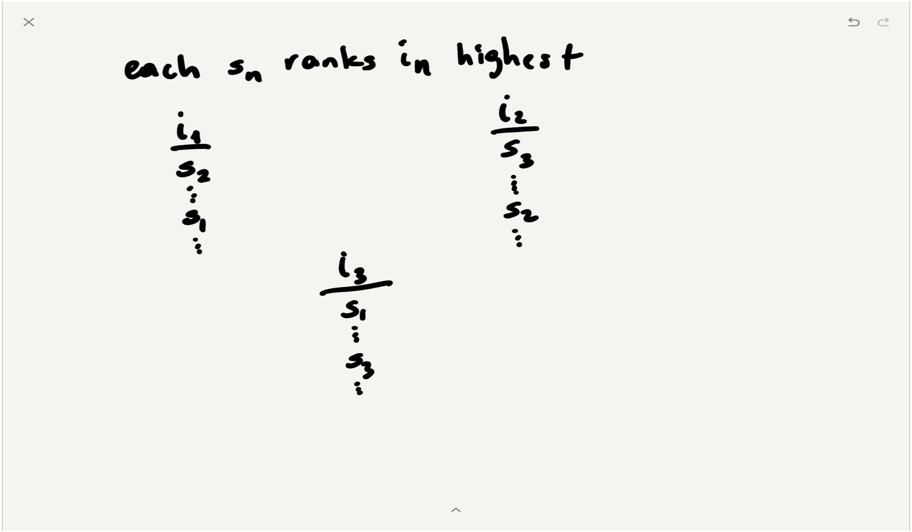
Theorem

Suppose each school has one seat. Let φ be a Pareto efficient and strategy-proof mechanism. If φ has less justified envy than TTC at $\succ$, then $\varphi(\cdot, \succ) = TTC(\cdot, \succ)$.

Corollary

Suppose each school has one seat. TTC is **justified envy minimal** *in the class of Pareto-efficient and strategy-proof mechanisms.*

Proof



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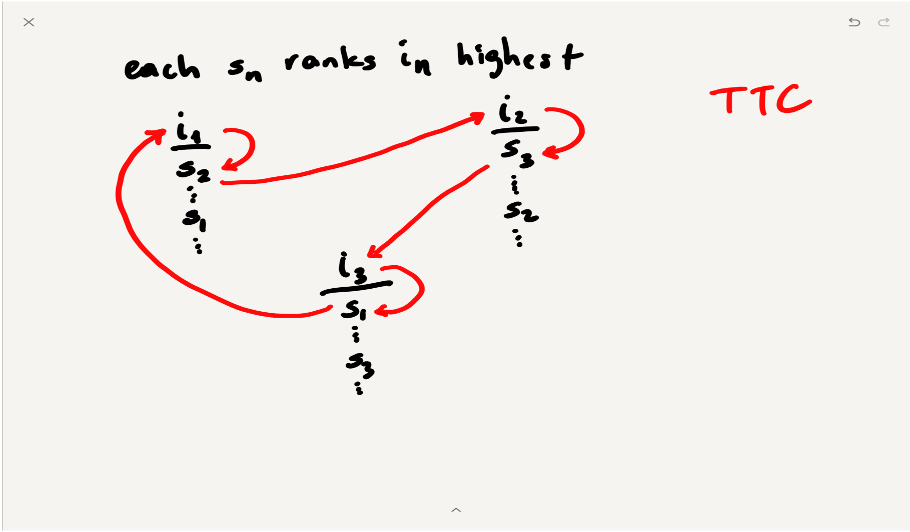

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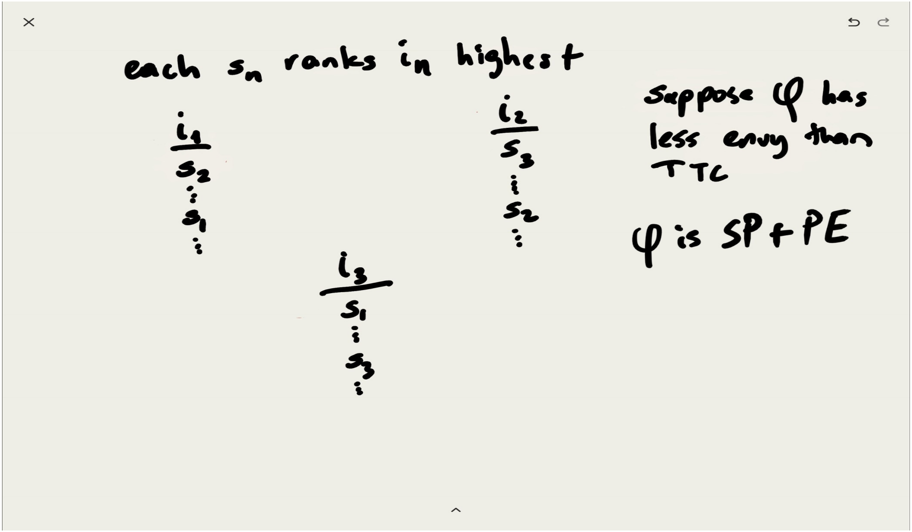

each s_n ranks i_n highest
i_1
s_2
s_1
i_2
s_3
s_2
i_3
s_1
s_3
Suppose φ has less envy than TTC
φ is SP + PE

Proof

each s_n ranks i_n highest

i_1	i_3	i_2
s_2	s_1	s_3
$\vdots$	$\vdots$	$\vdots$
s_1	s_3	s_2
$\vdots$	$\vdots$	$\vdots$

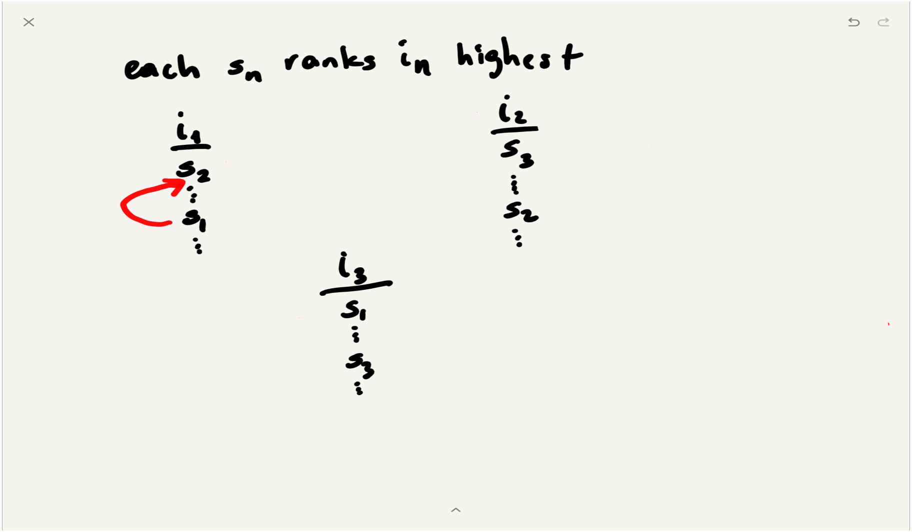
Suppose φ has less envy than TTC

suppose $\varphi(i_1) \neq TTC(i_1)$

then

i_1 prefers TTC outcome

Proof



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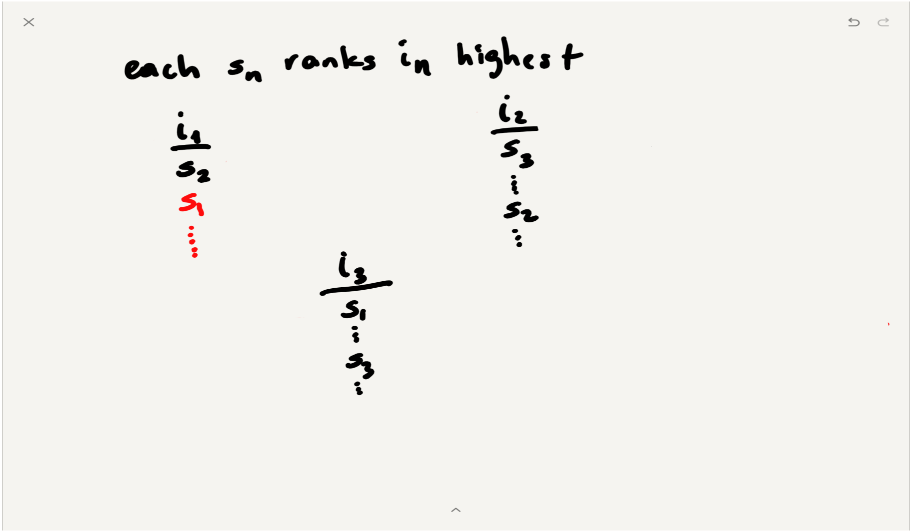


Proof

each s_n ranks i_n highest

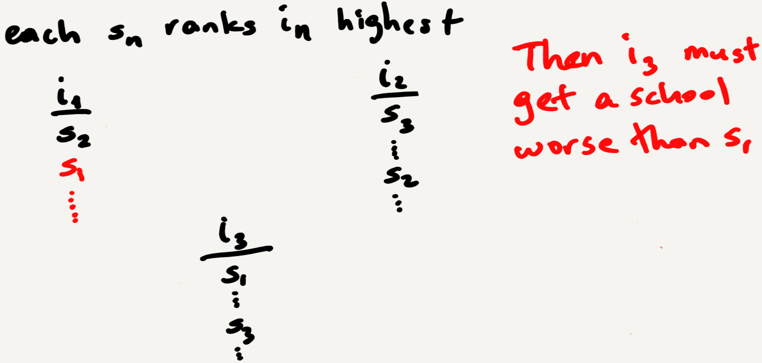
i_1: s_2, s_1, ⋮

i_2: s_3, ⋮, s_2, ⋮

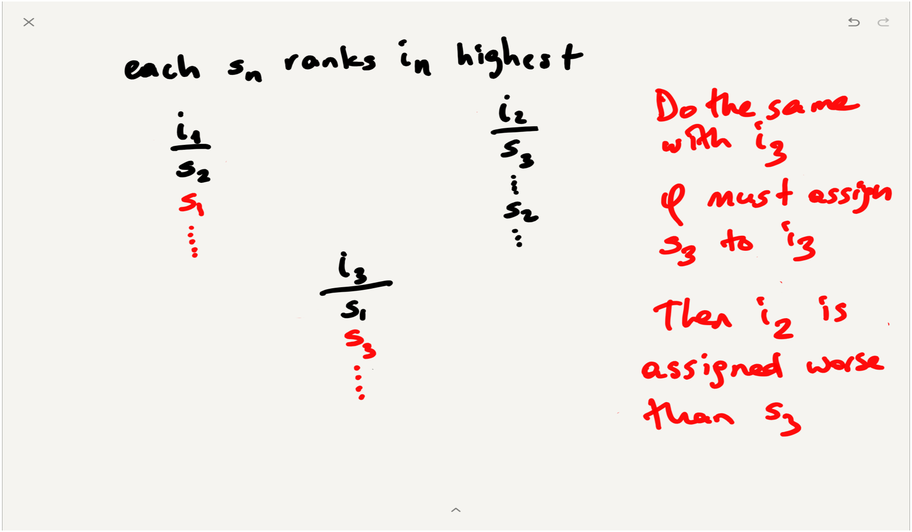
i_3: s_1, ⋮, s_3, ⋮

- TTC is same
- by SP, φ cannot assign s_2 to i_1
- So φ must assign s_1
- otherwise φ would not have less envy

Proof


each s_n ranks i_n highest
i_1
s_2
s_1
i_2
s_3
s_2
i_3
s_1
s_3
Then i_3 must get a school worse than s_1

Proof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s_1
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⋮
s_2
⋮
i_3
s_1
s_3
⋮
Do the same with i_3
& must assign s_3 to i_3
Then i_2 is assigned worse than s_3

Proof

each s_n ranks i_n highest

i_1
s_2
s_1
⋮

i_3
s_1
s_3
⋮

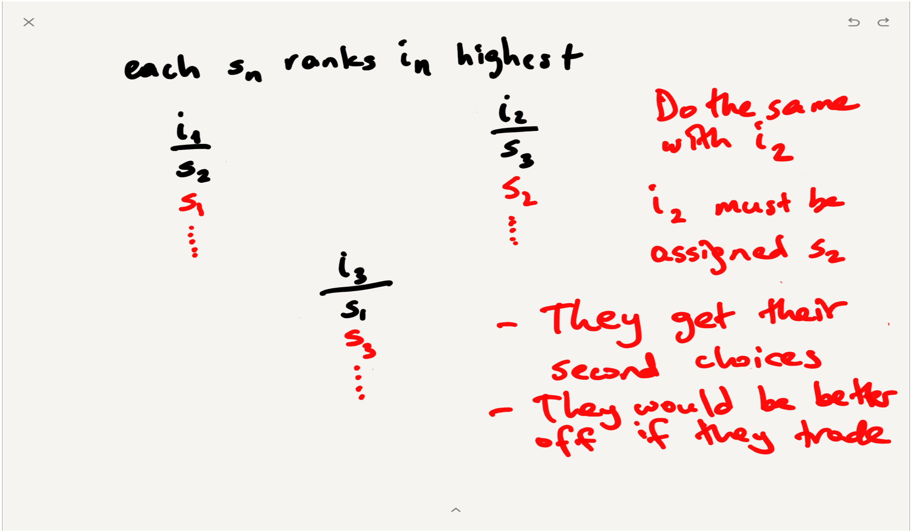
i_2
s_3
s_2
⋮

Do the same with i_2

i_2 must be assigned s_2

- They get their second choices

Proof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Do the same with i_2
i_2 must be assigned s_2
- They get their second choices
- They would be better off if they trade

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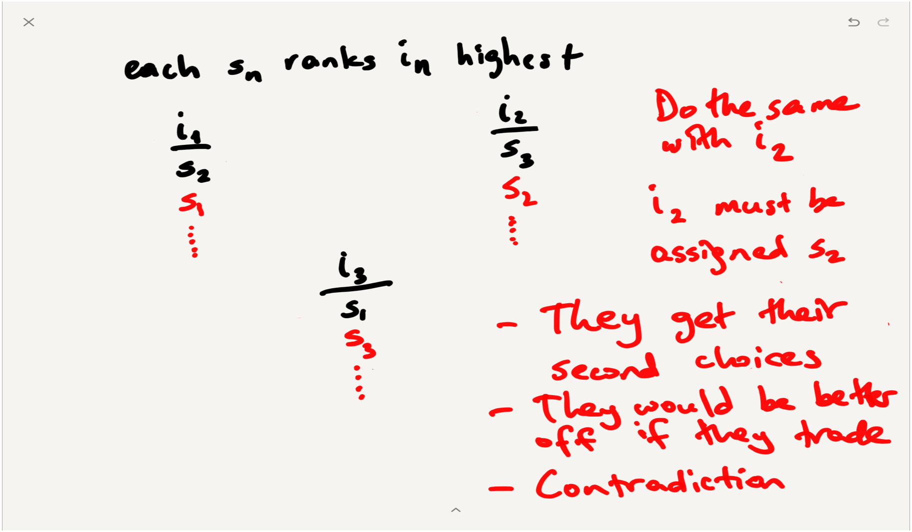
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Do the same with i_2
i_2 must be assigned s_2
- They get their second choices
- They would be better off if they trade
- Contradiction

Discussion

- One cannot further reduce justified envy in TTC without sacrificing efficiency or strategy-proofness.
- This result does not imply that TTC is the only justified-envy minimal mechanism in the class of Pareto efficient and strategy-proof mechanisms.
- However, consider a general class of SP+PE mechanisms:
  - $f_s :\succ_s \rightarrow \succ_s$ be an arbitrary function that transforms its priority into another (possibly same or distinct) priority. Let $f = (f_s)_s$.
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  - Let $\varphi(\cdot, \succ) = TTC(\cdot, f(\succ))$
  - Serial dictatorship is in this class
- Then,

Theorem

Suppose $f_s(\succ_s) \neq \succ_s$ for some school s. Then, the mechanism $\varphi(\cdot, \succ) = TTC(\cdot, f(\succ))$ is not justified-envy minimal.

Many-to-one

The three TTC variations

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- TTC-Counters, TTC-Clinch and Trade and Equitable TTC are not generally comparable in terms of justified envy in general, since the set of blocking pairs are non-empty and disjoint for both mechanisms.
- Furthermore

Theorem

Suppose there is a school with more than one seat. Then, for each of the three mechanisms, there exists a justified-envy minimal, Pareto efficient, and strategy-proof mechanism that has strictly less justified envy than that mechanism.

Back to the more basic question

- Can we compare TTC and RSD?
- In general, TTC does not have less justified envy than RSD at some profile of priorities.
- How can we capture our intuition that TTC has less envy than RSD? On average/in expectation?

Example with $n = 2$: Relevance of Priorities

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  - But under RSD, the assignment is random, so there is $1/2$ chance of "justified envy."
- NB: Well-known equivalence result (originally by Abdulkadiroglu-Sonmez, but more appropriately Pathak-Sethuraman (2011)) does not apply to the joint distribution of assignment.

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- TTC and RSD give rise to identical ex ante assignments
- However, the likelihood with which justified envy arises is different under two mechanisms:
- For mechanisms $\mathcal{M} = \{TTC, RSD\}$ and any pair (i, s) where i is assigned lower than s, $N^{\mathcal{M}}(i, s)$ is the number of students assigned to school s with lower priority.

Random Priorities – TTC vs Random Serial Dictatorship

Theorem

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Given a student-school pair (i, s)*,* $N^{RSD}(i, s)$ *first-order stochastically dominates* $N^{TTC}(i, s)$*.*

In addition, if, under TTC, student i *prefers school* s *to his assignment with strictly positive probability, then* $N^{RSD}(i, s)$ *strictly stochastically dominates* $N^{TTC}(i, s)$*.*

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The expected number of students with justified envy, the expected number of blocking pairs, and the expected number of students each student justifiably envies are all smaller under TTC than under RSD.

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TTC has "probabilistically" strictly less justified envy than RSD.

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- Since short cycle occurs with positive probability, TTC induces probilistically less justified envy than RSD.

Comparing Mechanisms in New Orleans

Table 1. Comparison of Mechanisms in New Orleans for Main Transition Grades (PK and Grade 9)

	TTC-Counters (1)	TTC-Clinch and Trade (2)	Equitable TTC (3)	Serial Dictatorship (4)	Student-Proposing Deferred Acceptance (5)
			A. Choice Assigned		
1	772	770	771	777	762
2	126	129	127	121	137
3	46	47	47	44	51
4	18	18	18	17	19
5+	11	11	11	8	10
Unassigned	222	221	222	228	217
Total	1196	1196	1196	1196	1196
			B. Statistics on Blocking Pairs		
Students with justified envy	158	157	159	213	0
Schools involved in blocking pairs	7	7	7	12	0
Blocking pairs (i,s)	228	224	215	308	0
Instances of justified envy (i, (j,s))	1111	1086	1100	6546	0

Comparing Mechanisms in Boston

Table 2. Comparison of Mechanisms in Boston for Main Transition Grades (K1, K2, 6, and 9)

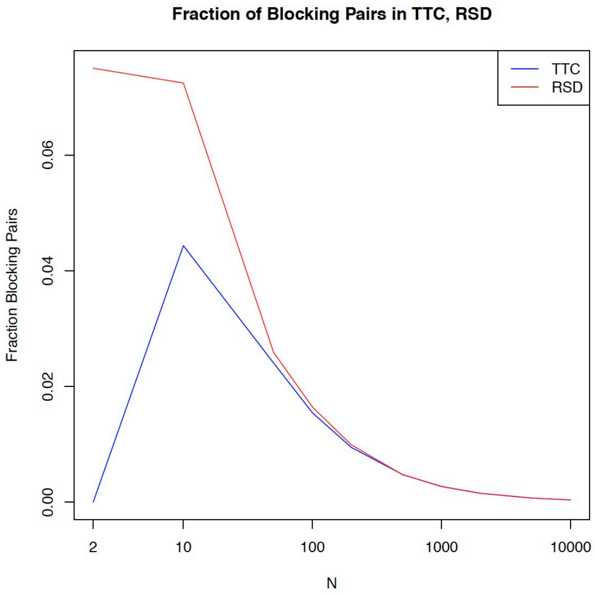
	TTC-Counters (1)	TTC-Clinch and Trade (2)	Equitable TTC (3)	Serial Dictatorship (4)	Student-Proposing Deferred Acceptance (5)
			A. Choice Assigned		
1	1240	1240	1240	1236	1227
2	322	323	323	315	336
3	134	134	134	132	138
4	56	55	55	51	57
5+	39	39	53	34	40
Unassigned	102	101	101	124	96
Total	1893	1893	1893	1893	1893
			B. Statistics on Blocking Pairs		
Students with justified envy	129	126	125	280	0
Schools involved in blocking pairs	18	18	18	44	0
Blocking pairs (i,s)	160	156	157	369	0
Instances of justified envy (i, (j,s))	768	711	696	3650	0

Large Markets

Large markets

- Boston and New Orleans have small number of schools
- What can we expect in environments with larger number of schools (very much like NYC)?
  - random priorities and random preferences
  - n individuals and schools (one-to-one) where $n \to \infty$

Example with large n: Asymptotic Irrelevance of Priorities



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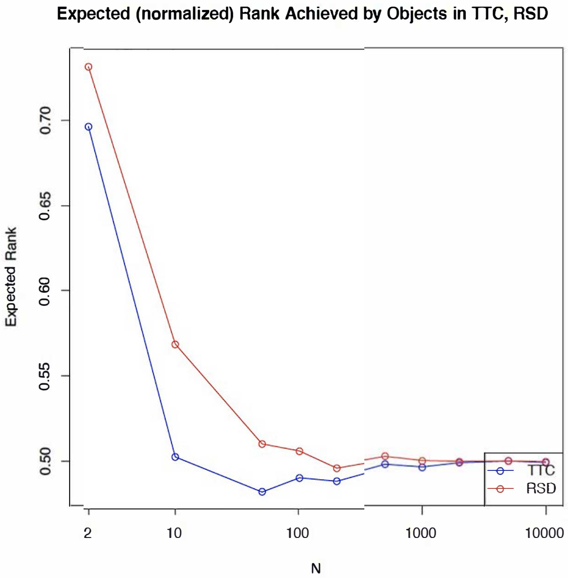

General Result: Asymptotic Irrelevance of Priorities

Let $\bar{R}_k$ be the normalized rank [i.e., rank/n] enjoyed by k (either an agent or a school) under TTC.

Theorem

$\{(\bar{R}_{s_j})_{j=1}^n, (\bar{R}_{i_k})_{k=1}^n\}$ converges in distribution to $\{(U[0,1])^n, (\bar{R}_{i_k})_{k=1}^n\}$ as $n \to \infty$, where $U[0,1]$ is uniform $[0,1]$.

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Corollary

In the limit, TTC induces the same distribution of the ranks enjoyed by all agents and schools, and thus the same incidence of justified envy, as RSD.

Intuition for the Asymptotic Irrelevance

- Key Observation 1: Let R_s^* be the priority rank of the agent that s points to when it is assigned under TTC. Then, the priority rank s enjoys under TTC $\sim U\{R_s^* + 1, ..., n\}$ conditional on the school being assigned via a long cycle.

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- Key Observation 2:
  (a) The proportion of schools assigned via long cycles $\xrightarrow{p} 1$.
  (b) The proportion of schools s for which $R_s^* < \log(n) \xrightarrow{p} 1$.

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- In words, virtually all schools are assigned via long cycles and point to very high priority individuals (in relative ranks) when assigned.

$\Rightarrow$ In a sufficiently large market, the schools' normalized ranks are each distributed according to $U[0, 1]$, independently of the ranks enjoyed by the agents.

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- To obtain (a), we characterize the probabilistic structure of TTC
  (i) the number of agents and schools follow a simple Markov chain: not trivial due to the conditioning issue.
  (ii) the number of rounds required for TTC is sublinear in n.
  (iii) the expected number of schools assigned via short cycles per round is bounded (by 2).

$\Rightarrow$ Combining (ii) and (iii) give (a)

- To obtain (b), we imagine a new mechanism TTC*—same as TTC except that schools are assigned agents that *schools point to*. Pareto efficiency from schools' perspective leads to (b) [see Che and Tercieux (TE, 2018)].

Robustness

- Correlated Preferences:
  - Weak correlation: $u_i(o) = u_s + \xi_{is}$, where u_s and ξ_{is} have bounded support. Then, our result is robust.
  - Extreme correlation: All have identical preferences. *TTC* admits fewer justified envy than does *RSD*.
- Other asymptotics:
  - "Large school" asymptotics: finite number of schools (Abdulkadiroglu-Che-Yasuda, 14; Azevedo-Leshno, 16; Leshno-Lo, 17): irrelevance does not hold since short cycles do not vanish.
- *Ultimately clarifies when priorities are relevant under TTC in a large market setting.*

Thank you!