Finite Horizon Asynchronous Games with Transfers: A Welfare Theorem*

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Abstract

This paper - and its companion piece, Dutta-Siconolfi (2016a) - prove a First Welfare Theorem for Games. It is shown that finite horizon asynchronous games with voluntary one period ahead transfers have a unique equilibrium that coincides with the Utilitarian Pareto Optimum. Whilst it is commonly thought that Folk Theorems are endemic in dynamic games, the result relies critically on two assumptions - simultaneous moves and no transfers. Yet asynchronicity and transfers are observed in many applications. We also show that the actions and transfers implied by our result are Markovian and easy to compute.

1 Introduction

The most important and most cited result in dynamic games is the Folk Theorem. By now, there are over a hundred papers that prove the result.¹ All these papers have one common assumption: simultaneous moves.

Though the simultaneous move model is a reasonable model in many contexts, it is unclear why it should be the only benchmark within which to study repeated interaction (especially since, thanks to the Folk Theorem, the model has no predictive power). Clearly, there are many applications in which players move at different times and they do know what their opponents last did. Indeed, many seminal papers have made a case for non-simultaneity. In bargaining, Rubinstein (1982) has pointed out that offers are followed by counter-offers. Maskin and Tirole (1988a, b) have argued that firms are sometimes committed to an action due to exogenous or technological reasons, e.g., they may have installed capital that has little scrap value, or there may be lags in producing

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¹The result has been proved for finite and infinite horizon, infinitely lived players, overlapping generations, complete and incomplete information, public and private monitoring...
and disseminating price lists. Alternatively, there may be short-term contracts that bind a firm temporarily to an action. In these cases, Maskin and Tirole argue, a competitor is able to react before the commitment expires, i.e., players effectively alternate their moves.2

These authors have shown that alternating moves can have interesting equilibrium consequences. Rubinstein (1982) showed that, unlike in simultaneous offer bargaining where any share is a static equilibrium, there is a unique equilibrium when players alternate offers (and it asymptotically replicates the Nash Bargaining Solution). Maskin and Tirole (1988a, b) showed that classic Industrial Organization predictions like the kinked demand curve or the Edgeworth cycle could emerge as equilibria when firms make short-term commitments.

Yet these results are still special in that they are one of many equilibrium possibilities. Asynchronous games are examples of stochastic games (with the fixed actions in a period being the "state"). Hence, by Dutta (1995) - for the infinite horizon - and Marlats (2009) - for the finite horizon - there is a Folk Theorem in these games as well. It appears then that with repeated interaction we are doomed to infinite multiplicity and consequent lack of predictive power.

This paper shows things are much cheerier if we make one additional assumption. If, in addition to committing to an action for one period, as in Rubinstein or Maskin and Tirole, players can also commit to a transfer to be made to the next period’s mover3, then there is a Welfare Theorem: there is a unique equilibrium and that equilibrium is efficient; it coincides with the Utilitarian Pareto Optimum. Whilst this paper proves the result for the finite horizon, the companion does so for the infinite horizon.

Note that in static game theory transfers are widely studied - Mechanism Design and Auction Theory, to name just two fields.

In the real-world there are many instances of non-simultaneity: collective bargaining4, time-stamping/encryption5, announcements of national policy6 etc.. Transfers are also widespread. In the Paris Climate Accord, there is a plan - the "Green Climate Fund" - to provide US$100 billion a year in aid to developing countries to implement new procedures to minimize climate change (with higher amounts in subsequent years). Private companies routinely make transfers: they pay mineral royalties to extract resources7, they pay licensing fees to

2Other papers that have studied asynchronous moves include Lagunoff and Matsui (1997) and Dutta (2012) in coordination, Admati and Perry (1987) and Marx and Matthews (2000) in public goods and the vast literature in evolutionary games that study birth-death processes.
3Note the class of transfers is restrictive in that they can be made conditional on the action chosen in the very next period only.
4Collective bargaining accounts for trillions of dollars on an annual basis in the United States. At the Federal level alone, the President and Congress negotiate an annual spending authorization of around $4 trillion.
5Timestamping establishes priority based on when messages are sent to a common pool and is widespread in encryption, including in Bitcoins.
6For example, the Paris Accord of 2015 was arrived at in a "bottoms up" way by individual countries asynchronously submitting "nationally determined contributions" (INDCs), in contrast to past attempts - like Kyoto - where there has been an (unsuccessful) attempt to simultaneously determine each country’s commitments.
7From mining companies, the US Government earns between $10bn and
use patented technologies\textsuperscript{8}, they pay copyright fees, trademark royalties, etc.

Most of this paper analyzes a two-player finite horizon alternating move game - though in Section 5 we show the same result holds for any $N$ player, asynchronous game. There is a fixed stage game (and no restriction is placed on it). In each period, only one player moves (and the other’s move remains fixed): she chooses an action from the stage game and a non-negative transfer schedule. Period payoffs for the mover are the sum of stage-game payoffs and the transfer from her opponent (according to the schedule picked in the previous period). Period payoffs for the non-mover are stage-game payoffs less transfer.

We show that there is a unique SPE and, in that equilibrium, actions coincide with the Utilitarian Solution - actions that a "planner" would prescribe if she were to maximize the sum of players’ payoffs over the game horizon. The uniqueness result is not particularly surprising given that we are studying a game of perfect information. The strong prediction is the efficiency result; no matter what stage game is played, players will always play the Utilitarian Solution. The associated equilibrium transfers are bounded.

We also compute transfers in the canonical stage games - Prisoners Dilemma and Battle of the Sexes.\textsuperscript{9} In the examples, transfers are either "traded" (with a player paying what he received in the previous period) or they are zero on the equilibrium path (since their use "off-equilibrium" is sufficient incentive for players to play efficiently on-equilibrium). Consequently, for long horizons, even inclusive of transfers SPE values are arbitrarily close to the Utilitarian values.

In the related literature \textit{either} asynchronicity has been studied or (static) transfers but not both. Yet one of the two assumptions is not enough. With only asynchronicity, as discussed above, by Dutta (1995) and Marlats (2009) there is a Folk Theorem because these are special cases of stochastic games.\textsuperscript{10}

What of transfers? For instance, what if the Prisoners Dilemma is played simultaneously but, ahead of play and simultaneously, players pick transfer schedules? There is a literature on Public Goods (Guttman (1978), Varian (1994)) that showed that one could indeed get the Pareto optimal outcome by this two-stage procedure. However, Jackson and Wilkie (2005) pointed out that those results are crucially predicated on the restriction that transfers can be made dependent only on the opponent’s action but not on one’s own. Once a richer set of transfer schedules is allowed, Jackson and Wilkie show that there are going to be multiple equilibria and none of them need be efficient.\textsuperscript{11}

\textsuperscript{8}$15bn a year in mineral leases. See https://cdn.americanprogress.org/wp-content/uploads/2015/06/RevenueOilGas-brief.pdf.
\textsuperscript{9}The size of these royalties can be quite significant; average royalty rates are about 15\% of EBITDA. See, Kemmerer and Lu (2012).
\textsuperscript{10}These are canonical in that they contain the only two equilibrium possibilities for finite horizon simultaneous move games. In the Prisoners Dilemma, uniqueness of the stage Nash Equilibrium implies uniqueness of SPE for all horizons. In the Battle of the Sexes, multiplicity of stage NE implies the Folk Theorem.
\textsuperscript{11}If the horizon is finite, the equilibrium will be generically unique but inefficient.
\textsuperscript{12}This paper considers the same set of transfer schedules as Jackson and Wilkie. In our setting, a mover at $t$ decides on a transfer schedule for $t+1$ (with the actual transfer depending on the opponent’s action choice in that period). Typically, the schedule chosen will also depend on the mover’s action choice at $t$. Hence, the transfer at $t+1$ will depend on both
Clearly, the assumption that players can commit to an action and a transfer schedule for one period is not a good assumption in many contexts. For instance, if other players' choices are unobserved, then moves should be modeled as being simultaneous. Our paper is inapplicable then.

On the other hand, Maskin and Tirole (1988a, b) have argued that action commitments are natural in some settings (see above). As for transfers, if there is an escrow account, then a commitment that will be honored can be made. Similarly, an announcement about a future transfer is credible if there are reputation costs to not paying; when countries make aid commitments they do honor those commitments. Finally, if there is a legal system that enforces contracts, then companies can sign contracts to make future payments. To summarize, our assumption that players can commit to a (one-period ahead) transfer is a reasonable assumption in some applications but may not be when legal systems are fragile or corporate entities are temporary. (In Section 5 we discuss what happens when this assumption is relaxed.)

Note though that the commitment is short-term, i.e., is only for one period. We are not allowing a player to commit to a life-time of transfers. That would turn our model into a one-period model and surely no one would be surprised at the efficiency result. The power of the result derives precisely from the fact that a minimal period commitment delivers a maximal period efficiency. Put differently, we propose a decentralized mechanism to implement the efficient solution, one in which players take turns being "Principals" and trade one-period contracts to generate dynamic efficiency.

In Section 2 we detail the model and in Section 3 prove the general theorem. Section 4 then computes equilibria in two representative stage-games - Prisoners Dilemma and Battle of the Sexes. In Section 5 we generalize the result to N players and any asynchronous game. Section 6 concludes. Some proofs and computations of secondary interest are in Section 7.

## 2 Model

Let G denote a two player stage game (in strategic form). (In Section 5 we extend the analysis to N players.) Denote player i's strategy set Ai and her payoff function πi, where, as usual, πi : A → R, and A ≡ A1 × A2 is the set of strategy tuples for the players. Suppose that Ai is finite for every i.\(^\text{12}\) Let i denote the generic player and let j denote the "other" player, j ≠ i.

The timing structure is that of alternating moves with players moving sequentially. Hence, player i moves in period t followed by player j in period t + 1 and so on. Whenever it is player i's turn to move, she can choose an action ait from Ai - and this action then remains fixed till t + 2. Additionally, a player can also make a conditional transfer to the other player - conditional in that it can be tied to the next period action of the other player.

\(^{12}\)The analysis extends to the case of Ai compact.
Payoffs are ongoing. If \( a_j \) denotes the (fixed) action of the other player, then the period \( t \) stage game payoffs are \( \pi_i(a_{it}, a_j), \pi_j(a_{it}, a_j) \). Additionally, if in the previous period the then mover (player \( j \)) had promised the current mover that she would be paid according to the schedule, \( \theta_j(a_{it}, a_j) \), then the total payoffs of player \( i \) inclusive of transfers will be \( \pi_i(a_{it}, a_j) + \theta_j(a_{it}, a_j) \) while that of player \( j \) would be \( \pi_j(a_{it}, a_j) - \theta_j(a_{it}, a_j) \).

There is an initial action state \( \pi \) for the game and that can be any action (of the player who doesn’t move in the initial period \( t = 0 \)). The initial transfer schedule for the game \( \theta(\cdot) \) is chosen at \( t = -1 \) before the game begins (by the player who doesn’t move in the initial period).\(^{13}\)

The horizon \( T \) is finite; lifetime payoffs are evaluated according to the undiscounted sum:\(^{14}\)

\[
\sum_{t=0}^{T} \{[\pi_i(a_{it}, a_{jt-1}) + \theta_j(a_{it}, a_{jt-1}) \parallel i \ \text{mover}] + [\pi_i(a_{it-1}, a_{jt}) - \theta_i(a_{jt}, a_{it-1}) \parallel j \ \text{mover}]\}
\]

### 2.1 Strategies and Equilibrium

#### Informational Assumption - All stage game actions and past transfer commitments are observable.

Hence, an action history \( h_i^t \) at time \( t \) includes all the past actions, \( h_i^t = (\pi, a_{j0}, a_{i1}, \ldots, a_{jt-1}) \). Similarly, a transfer history \( h_j^t \) includes all past transfer commitments - \( h_j^t = (\theta(\cdot), \theta_{j0}(\cdot), \ldots, \theta_{jt-1}(\cdot)) \).\(^{15}\) To summarize, a history \( h_t \) is given by \( h_t = (h_i^t, h_j^t) \).

Note also that for the \( t - th \) period action choice \( a_{it} \), the payoff relevant parts of the history are only the preceding period’s choices: \( a_{jt-1}, \theta_{jt-1}, \) the fixed action of the other player \( j \) and the transfer schedule that \( j \) has picked. For the \( t - th \) period transfer choice \( \theta_{it}(\cdot) \), which is only paid in period \( t + 1 \), there is no payoff relevant part of the history \( h_t \) since the \( t + 1 \) period payoffs will depend on \( a_{it} \) and \( a_{jt+1} \) neither of which are in \( h_t \).

A \( t - th \) period strategy \( \sigma_{it} \) for player \( i \) is an action choice \( a_{it} \) that maps from a history \( h_t \) and a transfer choice \( \theta_{it} \) that maps from \( (h_t, a_{it}) \). In the usual fashion, a strategy for player \( i \) in the game, \( \sigma_i \), is a specification of a strategy \( \sigma_{it} \) in every period that she is the mover. A strategy vector - one strategy for every player - defines a (possibly probabilistic) history \( h_t \). Denote the lifetime payoff of the mover at time \( t \), with, say \( \tau = T - t \) periods left in the game, \( V_{it} \):

\[
V_{it}(h_t) = \max_{a_{it}} \{[\pi_i(a_{it}, a_{jt-1}) + \theta_{jt-1}(a_{it})] + W_{i(t-1)}(h_t, a_{it})\} \tag{1}
\]

\(^{13}\)The initial transfer - like the initial action - can, in principle, be arbitrary. The optimality results that follow will then hold after the first period. Alternatively, both initial action state and transfer schedule may be chosen in a "pre-period", period \(-1\), before the game begins. That, for instance, is the way Rubinstein (1982) works.

\(^{14}\)The analysis extends to the discounted case.

\(^{15}\)Note that the transfer history includes information about the entire transfer schedule and not just the actual transfer that was made in a previous period. Any results that we prove will continue to hold if we replaced that requirement with the weaker one that said only the actual transfers are observable. From now on to simplify notation we write \( \theta \) for \( \theta(\cdot) \).
where $W_{i\tau-1}$ is the continuation for player $i$ that follows from the optimal choice of a transfer schedule $\theta_{it}$, given $a_{it}$, and anticipating the "Stackelberg follower" player $j$'s best response $a_{jt+1}$ to that transfer, i.e.,

$$W_{i\tau-1}(h_t, a_{it}) = \max_{\theta_i} \{ \pi_i(a_{it}, a_j(\theta_i)) - \theta_i(a_j) \} + \delta V_{i\tau-2}(h_{t+2}(a_{it}, \theta_{it}))$$

where $h_{T+2}(a_{it}, \theta_{i\tau})$ is the history at period $t+2$ caused by player $i$'s period $t$ actions.

A Subgame Perfect Equilibria (SPE) is a pair of strategies that are best responses to each other. SPE will be computed by backwards induction. Markov Strategies are choices that depend only on the payoff-relevant histories and the number of periods $\tau$ remaining. A Markov Perfect Equilibria (MPE) is a pair of Markov strategies that are best responses to each other. Backwards induction generates MPE. As is well-known, MPE are also SPE.

### 3 Main Theorem

Let us start with a benchmark - the Utilitarian Pareto Optimum:

**Definition 1** The Utilitarian Pareto optimum (UPO) problem is the maximization of the sum of players payoffs subject to an initial condition specifying a fixed action of the first period non-mover:

$$\max_{\{a_{it}, a_{jt}\}_{t\geq 0}} \sum_{t=0}^{T} \{ \pi_i(a_{it}, a_{jt}) + \pi_j(a_{it}, a_{jt}) \}$$

s.t. $a_{i0} = \bar{a}_i$, $a_{it} = a_{it-1}$, $t \geq 1$, $t$ odd, and $a_{jt} = a_{jt-1}$, $t \geq 0$, $t$ even

It is well known that the solution is Markovian, i.e., there is $\hat{\pi}_r(a_i)$ - respectively, $\hat{\pi}_r(a_j)$ - such that the Pareto optimal solution, with $\tau$ periods to go, is for player $i$ to play according to $\hat{\pi}_r(a_i)$ - respectively for player $j$ to play $\hat{\pi}_r(a_j)$ - whenever it is her turn to move.

The main theorem that ties equilibrium behavior to the UPO is:

**Theorem 2** There is a unique SPE. It coincides with the Utilitarian Pareto optimum solution in terms of actions - with $\tau$ periods to go, the equilibrium is for player $i$ to play according to $\hat{\pi}_r(a_i)$ - respectively for player $j$ to play $\hat{\pi}_r(a_j)$ - whenever it is her turn to move. The transfer given makes the mover as well off as she would be from picking her most preferred action in the absence of current transfers.

**Proof** For purely notational reasons, suppose that the last player to move is player $j$. For any $a_i$, define the one-period utilitarian maximum:

$$\hat{\pi}_1(a_i) = \arg \max_{a_j} \{ \pi_i(a_i, a_j) + \pi_j(a_i, a_j) \}$$
and let the associated one-period utilitarian maximum value be denoted by

\[ U_1(a_i) = \pi_i(a_i, \sigma_1(a_i)) + \pi_j(a_i, \sigma_1(a_i)) \]

Inductively, whenever player \( j \) is the mover, define the \( \tau + 1 \) period utilitarian maximum

\[ \sigma_{\tau+1}(a_i) = \arg \max_{a_j} \{ [\pi_i(a_i, a_j) + \pi_j(a_i, a_j)] + U_{\tau}(a_j) \} \]

and let the associated \( \tau + 1 \)-period utilitarian maximum value be denoted

\[ U_{\tau+1}(a_j) = \pi_i(a_i, \sigma_{\tau+1}(a_i)) + \pi_j(a_i, \sigma_{\tau+1}(a_i)) + U_{\tau}(\sigma_{\tau+1}(a_i)) \]

with analogous optimal action, \( \sigma_{\tau+1}(a_j) \) and utilitarian value, \( U_{\tau+1}(a_j) \) defined for periods in which player \( i \) is the mover.

Define now individual incentives in the following manner. For any \( a_i \) define the one-period unilateral maximum:

\[ \sigma_1^*(a_i) = \arg \max_{a_j} \pi_j(a_i, a_j) \]

and let the associated one-period unilateral maximum be denoted

\[ V_1^*(a_i) = \pi_j(a_i, \sigma_1^*(a_i)) \]

By the BIR principle, for every action \( a_i \) we can define the transfer required to induce one-period play of \( a_j \) as

\[ \theta_1(a_j|a_i) = V_1^*(a_i) - \pi_j(a_i, a_j) \quad (4) \]

Clearly, \( \theta_1(a_j|a_i) \geq 0 \) for all \( a_j \). Define the non-mover’s, player \( i \)'s, payoff in the last period as

\[ W_1(a_i) = \max_{a_j} [\pi_i(a_i, a_j) - \theta_1(a_j|a_i)] \]

Substituting for \( \theta_1(a_j) \) from Eq. 4, it follows that the maximization above is equivalent to

\[ W_1(a_i) = \max_{a_j} [\pi_i(a_i, a_j) + \pi_j(a_i, a_j)] - V_1^*(a_i) = U_1(a_i) - V_1^*(a_i), \]

the OIM principle. Therefore, we have the following claim:

**Claim 3** When there is one period left in the game, regardless of the action \( a_i \) played in the immediately preceding period by (the non-mover) player \( i \), the unique equilibrium continuation is for \( i \) to offer a transfer \( \theta_1(\sigma_1(a_i)|a_i) \) to induce the play of the utilitarian maximum \( \sigma_1(a_i) \) in the last period. The associated payoffs for the mover, player \( j \), and the non-mover are then, respectively, the unilateral maximum value \( V_1^*(a_i) \) and the residual from the utilitarian maximum value \( U_1(a_i) - V_1^*(a_i) \).
Similarly when there are 2 periods left, for any \(a_j\), define the two-period unilateral maximum:

\[
\sigma_2^*(a_j) = \arg \max_{a_i} [\pi_i(a_i, a_j) + W_1(a_i)]
\]

and let the associated two-period unilateral maximum be denoted by

\[
V_2^*(a_j) = \pi_i(\sigma_2^*(a_j), a_j) + W_1(\sigma_2^*(a_j))
\]

Again by BIR principle, for every action \(a_i\) define the transfer required to induce penultimate-period play of \(a_i\) as

\[
\theta_2(a_i | a_j) = V_2^*(a_j) - [\pi_i(a_i, a_j)] + W_1(a_i)]
\]

Clearly, \(\theta_2(a_i | a_j) \geq 0\) for all \(a_i\). Define the non-mover’s, player \(j’\)s, continuation payoff starting in the penultimate period as

\[
W_2(a_j) = \max_{a_i} [\pi_j(a_i, a_j) - \theta_2(a_i | a_j) + V_1^*(a_i)]
\]

It follows from OIM, substituting for \(\theta_1(a_i | a_j)\) from Eq. 5 that the maximization above is equivalent to

\[
W_2(a_j) = \max_{a_i} [\pi_i(a_i, a_j) + \pi_j(a_i, a_j) + W_1(a_i) + \nu_1^*(a_i)] - V_2^*(a_j) = U_2(a_j) - V_2^*(a_j)
\]

so that we have the following claim:

**Claim 4** When there are two periods left in the game, regardless of the action \(a_j\) played in the third to last period by current non-mover \(j\), the unique equilibrium continuation in the penultimate period is for \(j\) to offer a transfer \(\theta_2(\sigma_2(a_j) | a_j)\) to induce the play by \(i\) of the utilitarian maximum \(\sigma_2(a_j)\) in the second to last period. The associated payoffs in the last two periods for the mover and non-mover are then, respectively, the unilateral maximum value \(V_2^*(a_j)\) and the residual from the utilitarian maximum value \(U_2(a_j) - V_2^*(a_j)\).

Inductively define, for periods when player \(i\) is the mover, optimal action, \(\sigma_{T+1}(a_j)\) and utilitarian value, \(U_{T+1}(a_j)\). Similarly, for periods when player \(j\) is the mover, define an optimal action, \(\sigma_{T+1}(a_i)\) and utilitarian value, \(U_{T+1}(a_i)\). The same proof as above applies to give us the main result.

Note that on path transfers are pinned down by the above construction, while off equilibrium transfers are indeterminate as all they need to do is satisfy conditions 4, 5 (and their equivalent for \(\tau > 2\)) as inequalities.

## 4 Two Examples

In this section, we present two examples - Prisoners Dilemma and Battle of the Sexes. As discussed in the Introduction, the two are representative of the only two possibilities that can arise with the canonical finitely repeated (simultaneous
move, no transfers) game. If the stage game has a unique Nash Equilibrium - as in the Prisoners Dilemma - then (by backward induction) repeated play of the stage Nash is the unique SPE of the finitely repeated game. However, that SPE can be - and is for the Prisoners Dilemma - sub-optimal. On the other hand, if the stage game has multiple Nash Equilibria - as in the Battle of the Sexes - then (by Benoît and Krishna (1985)) there is a Folk Theorem such that every individually rational payoff can emerge as a SPE payoff for long enough horizon. Here optimal payoffs can emerge as SPE but uniqueness is lost.

In our setting, consider the Prisoners Dilemma first. Using non-standard terminology for the stage-game actions, let us call \textit{bad} the dominant strategy in the stage-game and \textit{good} the optimal one. So, whereas in the standard model \((\text{bad, bad})\) always is the unique SPE, in our model \((\text{good, good})\) always is the unique SPE. We compute transfers - and it is straightforward to do so - and show that, inclusive of transfers, the average lifetime payoffs are arbitrarily close to the average UPO payoffs.

In the Battle of the Sexes game, we show that the Folk Theorem disappears. Recall that, in the Battle of the Sexes stage game, there are two Nash Equilibria and each is Pareto Optimal. If both are UPO solutions then the initial state of the game - the action that is initially fixed - selects which of these two stage game NE is absorbing. The SPE then mimics that - it too is initial state dependent and remains at the stage game NE that "starts" the game and never departs into a payoff destroying cycle.

The reader might wonder what happens if moves are alternating but no transfers are allowed. Put differently, what would be the consequence of making one but not two changes to the standard model. We explore that issue - partly because it is, pedagogically, a useful intermediate step. We show that in the Prisoners Dilemma, the conclusions are identical to those in the canonical model; again, there is a unique SPE which is the play of \textit{bad} always and after all histories. In the Battle of the Sexes, there is no longer a Folk Theorem but rather there is a unique SPE. (After all, the alternating move game is a game of perfect information and hence, generically, has a unique SPE.) However, that equilibrium does not coincide with the Utilitarian Pareto Optimum. In particular, in that equilibrium, each player tries to "steer" the game to her favorite stage game Nash Equilibrium, one of them always succeeds but, in doing so, destroys payoffs.

### 4.1 Prisoners Dilemma

Consider the Prisoners Dilemma stage game:

<table>
<thead>
<tr>
<th></th>
<th>bad</th>
<th>good</th>
</tr>
</thead>
<tbody>
<tr>
<td>bad</td>
<td>0, 0</td>
<td>(c, b)</td>
</tr>
<tr>
<td>good</td>
<td>(b, c)</td>
<td>(d, d)</td>
</tr>
</tbody>
</table>

with \(2d > b + c > 0\) and \(c > d\) and \(0 > b\). Note that \textit{bad} is the dominant strategy and the unique Utilitarian solution in the stage game is \((\text{good}, \text{good})\).
4.1.1 Simultaneous Play, No Transfers

**Stage Game:** There is a unique Nash Equilibrium in dominant strategies: 
(bad, bad) - with payoffs (0, 0). Those payoffs - (0, 0) - are also the min-max payoffs for each player.

**Finite Horizon Play:** By backward induction, as is well-known, there is a unique SPE with payoffs equal to (0, 0) and the associated strategies are to play (bad, bad) regardless of history. In particular, the unique SPE yields sub-optimal payoffs.

4.1.2 Alternating Moves, No Transfers

Suppose instead that the stage game is played with alternating moves but no transfers over a horizon \( T \).

**Utilitarian Pareto Optimum** Suppose we pick the mover’s action in each period so as to maximize \( \sum_{t=1}^{T} (\pi_{1t} + \pi_{2t}) \), subject to a given initial action ("state") such as good - or bad. It is not difficult to see that: The UPO action is to play good regardless of history. Hence, after at most one period, play is absorbed at (good, good).

**Finite Horizon SPE** Consider instead the SPE. The following results is a straightforward application of backwards induction (and hence not proved):

There is a unique SPE and the associated strategies are to play bad regardless of history. So, after at most one period, play is absorbed at (bad, bad).

4.1.3 Alternating Moves with Transfers

Clearly the UPO solution remains unchanged from the previous sub-section - play good regardless of history. We now compute the transfers required to implement the UPO solution and the associated payoffs. Let \( \tau \) denote the number of remaining periods and let \( V_{\tau}(g) \) and \( V_{\tau}(b) \) denote the total payoffs, inclusive of transfers, in the (unique) SPE. Similarly, let \( W_{\tau}(g) \) and \( W_{\tau}(b) \) denote the total payoffs, inclusive of transfers, over the remaining \( \tau \) periods for the non-mover. In each case, \( g \) stands for good as the initial state - and \( b \) for bad as the initial state.

**\( \tau = 1 \)** Since good is played in the last period after all histories in the UPO but bad is a dominant action, transfers are required. They can be computed as 
\[
\theta_1(g|b) = -b, \quad \theta_1(g|g) = c - d,
\]

where \( \theta_1(g|g) \) is the transfer to the mover in the last period when she plays good (and the fixed action is good) and similarly \( \theta_1(g|b) \) is the transfer to the mover in the last period when she plays good (and the fixed action is bad). Note the subscript refers to the fact that there is 1 period left in the game. The exact computation is simply derived from making the mover indifferent between the

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16 The sketch of the argument is that bad is the last period best response after all histories, thereby making bad the penultimate period best response after all histories ... And so on.
actions \textit{bad} and \textit{good}, i.e., apply the BIR principle. Of course, no transfers are made if the mover picks \textit{bad} as her action.

\(\tau = 2\) Suppose the fixed action is \textit{bad}. The transfer \(\theta_2(g|b)\) can be similarly derived from making the mover indifferent between the actions \textit{bad} and \textit{good}:

\[
c - \theta_1(g|b) = b + \theta_2(g|b) + d - \theta_1(g|g),
\]

(6)

Note that the LHS is the payoff over two periods if the current mover picks \textit{bad} (given fixed action \textit{bad}). That gives him 0 immediately. Thereafter, as seen from \(\tau = 1\) above, he induces the other player to play \textit{good} by paying the appropriate transfer all of which gets him a payoff of \(c - \theta_1(g|b)\) in the last period. The RHS - the two period payoff to playing \textit{good} today - is similarly derived from an immediate payoff of \(b + \theta_2(g|b)\) being added to a last period payoff of \(d - \theta_1(g|g)\). Again, no transfers are made if the mover picks \textit{bad} as her action. The above equation yields - after substituting for \(\theta_1(g|b)\) and \(\theta_1(g|g)\),

\[
\theta_2(g|b) = 2(c - d).
\]

Similarly, the transfer \(\theta_2(g|g)\) is derived from

\[
c + c - \theta_1(g|b) = d + \theta_2(g|g) + d - \theta_1(g|g),
\]

(7)

that is,

\[
\theta_2(g|g) = 3(c - d) + b,
\]

assuming that the latter is greater than 0.\footnote{We address the alternative case, \(3(c - d) + b < 0\) later.}

\(\tau = 3\) Suppose that the fixed action is \textit{bad}. The transfer \(\theta_3(g|b)\) is derived from

\[
c - \theta_2(g|b) + V_1(g) = b + \theta_3(g|b) + d - \theta_2(g|g) + V_1(g),
\]

(8)

where, \(V_1(g)\) is the last period payoff when the fixed action is \(g\), and the presence of \(V_1(g)\) on both sides of the equation is due to \textit{good} being absorbing within two periods. It is straightforward to check that \(\theta_2(g|g) - \theta_2(g|b) = \theta_1(g|g) - \theta_1(g|b)\). Hence, Eqs. 6 and 8 are identical and so have identical solutions:

\[
\theta_3(g|b) = \theta_2(g|b) = 2(c - d).
\]

Similarly, the transfer \(\theta_3(g|g)\) is derived from

\[
c + c - \theta_2(g|b) + V_1(g) = d + \theta_3(g|g) + d - \theta_2(g|g) + V_1(g).
\]

(9)

Again Eqs. 7 and 9 are identical and hence,

\[
\theta_3(g|g) = \theta_2(g|g) = 3(c - d) + b.
\]

In fact, for any \(\tau > 3\), the argument repeats. Make the induction hypothesis that \(\theta_i(g|b) = 2(c - d)\) for \(1 < t < \tau - 1\) and consider period \(\tau\). Similar equations as 8 and 9 are derived with the one difference that instead of \(V_1(g)\)
we have $V_{T-2}(g)$ (since good is absorbing within two periods). Hence, $\theta_\tau(g|b) = \theta_2(g|b)$ and $\theta_\tau(g|g) = \theta_2(g|g)$, for all $\tau$.

What if $3(c - d) + b < 0$? The derivation is very similar and so we simply sketch the argument here. As we will see, the one interesting departure is that transfers are no longer required once play is absorbed at good, i.e., $\theta_1(g|g) = 0$ for all $t$ other than the last period. So if the initial state is good, there is never a transfer paid till the last period! If the initial state is bad, there is exactly one transfer, and that in the initial period (again till the last period).

The analysis is identical for the last period, $\tau = 1$, and for $\tau = 2$, with fixed state bad. But at $\tau = 2$, if the fixed state good, it is now the case that, even with $\theta_2(g|g) = 0$, the RHS of Eq. 7 is greater than the LHS. Put another way, $\theta_2(g|g) = 0$, and no transfer is required to induce play of the Utilitarian solution at that history.

For $\tau = 3$, Eq. 8 consequently becomes

$$c - \theta_2(g|b) + V_1(g) = b + \theta_3(g|b) + d + V_1(g)$$

from which it follows by simple substitution that

$$\theta_3(g|b) = -b - (c - d)$$

which is positive under the hypothesis for this case, $-b > 3(c - d)$. What of $\theta_3(g|g)$? Eq. 9 is now

$$c + c - \theta_2(g|b) + V_1(g) = d + \theta_3(g|g) + d + V_1(g).$$

Since $\theta_2(g|b) = 2(c - d)$ it follows that $\theta_3(g|g) = 0$. For any $\tau > 3$, the argument by induction is identical to what we did above. All of this leads to:

**Proposition 5** There is a unique SPE whose actions mimic the UPO and always play good. Hence, after at most one period, play is absorbed at (good, good). In the last period transfers are, respectively $-b$ and $c - d$ following play of bad and good. In other periods, transfers are:

Case 1 - $3(c - d) + b > 0$: Regarding history, the transfer is $2(c - d)$ having played bad in the previous period, and $3(c - d) + b$ having played good. Inclusive of transfers, per period SPE payoffs are arbitrarily close to $(d, d)$: if the initial state is bad and $T$ is even, then $V_T(b) = d - \frac{b}{T}$ and $W_T(b) = d + \frac{b + 2(c - d)}{T}$ while if the initial state is good, then $V_T(g) = d + \frac{b + 2(c - d)}{T}$, $W_T(g) = d - \frac{b + 2(c - d)}{T}$.18

Case 2 - $3(c - d) + b < 0$: Having played bad in the previous period, in any period save the last two, the transfer is $-b - (c - d)$, while in the second-last period it is $2(c - d)$. The transfer is 0 having played good. Inclusive of transfers, per period SPE payoffs are again arbitrarily close to $(d, d)$: if the initial state is bad and $T$ is odd, then $V_T(b) = \frac{T - 1}{T} d$ and $W_T(b) = \frac{T - 1}{T} d + \frac{b + c}{T}$ while if the initial state is good, then $V_T(g) = d + \frac{c - d}{T}$, $W_T(g) = d - \frac{c - d}{T}$.19

---

18If $T$ is odd, and the initial state is bad, then $V_T(b) = d - \frac{2(c - d)}{T}$ and $W_T(b) = d + \frac{b + 2(c - d)}{T}$ while if the initial state is good, then $V_T(g) = d + \frac{2(c - d)}{T}$, $W_T(g) = d - \frac{2(c - d)}{T}$.19

19If $T$ is even, and the initial state is bad, then $V_T(b) = \frac{(T - 1)d - 2(c - d)}{T}$ and $W_T(b) = \frac{(T - 1)d + 2(c - d)}{T}$; if the initial state is good, then $V_T(g) = d - \frac{c - d}{T}$, $W_T(g) = d + \frac{c - d}{T}$.19
We finish this section by providing a numerical example. For instance, if \( c - d = 1 \), then we have Case 1, whenever \( b > -3 \), and Case 2 otherwise. For example, if \( b = -2 \), we are Case 1, and, starting at bad, a transfer of 2 gets play to good and it stays there with constant ("traded") transfers of 1 each period. On the other hand, if \( b = -4 \), we are Case 2, and, again starting at bad, a transfer of 3 gets play to good and thereafter there are no more transfers till the very last period (when there is a transfer of 1). So in both cases, for the "middle" \( T - 2 \) periods, transfers are zero across every pair of periods. Hence, average payoffs are arbitrarily close to \( d \).

4.2 Battle of the Sexes

Consider the Battle of the Sexes stage game:

<table>
<thead>
<tr>
<th></th>
<th>good</th>
<th>bad</th>
</tr>
</thead>
<tbody>
<tr>
<td>bad</td>
<td>( d, c )</td>
<td>0, 0</td>
</tr>
<tr>
<td>good</td>
<td>0, 0</td>
<td>( c, d )</td>
</tr>
</tbody>
</table>

with \( c > 2d > 0 \).

4.2.1 Simultaneous Play

**Stage Game:** As is well-known, when played simultaneously, there are two stage-game Nash Equilibria: (bad, good) - with payoffs \((d, c)\) - and (good, bad) - with payoffs \((c, d)\). Clearly, the former is preferred by player 2 while the latter is preferred by player 1. There is also a mixed-strategy Nash Equilibrium with payoffs \(\left( \frac{cd}{c+d}, \frac{cd}{c+d} \right)\). This Nash Equilibrium is hence not Pareto optimal.

Those payoffs - \(\left( \frac{cd}{c+d}, \frac{cd}{c+d} \right)\) - are also the *min-max payoffs* for each player. Hence, the (convex hull) of the set of Individually Rational Payoffs in the stage game is given by

\[ V^* = \left\{ (v_1, v_2) \geq \left( \frac{cd}{c+d}, \frac{cd}{c+d} \right) : (v_1, v_2) = \lambda(d, c) + \gamma(c, d), \ (\lambda, \gamma) \geq 0, \ \lambda + \gamma \leq 1 \right\}. \]

**Finite Horizon Play:** Suppose instead that the stage game is played (simultaneously) over a horizon \(T\). By Benoit and Krishna (1986) the following Folk Theorem result is known to be true: Take any \( v \in V^* \) and any \( \varepsilon > 0 \). There exists a horizon \( T(v, \varepsilon) \) such that whenever the game is played over horizon \( T \geq T(v, \varepsilon) \), there is a SPE with average payoffs within \( \varepsilon \) of \( v \).

4.2.2 Alternating Moves, No Transfers

**Finite Horizon Pareto Optimum** Suppose we pick the mover’s action in each period to maximize \( \sum_{t=1}^{T} (\pi_{1t} + \pi_{2t}) \), subject to some given initial action ("state") such as good - or bad. Clearly, in any period \( t \), the most that \( \pi_{1t} + \pi_{2t} \) can be is \( d + c \). In particular then, if the initial state is good and the first period mover is Player 1, then the UPO is to pick bad - thereby yielding \( \pi_1 + \pi_2 = c + d \).
- and follow that up by picking good for Player 2, followed by bad for Player 1 etc. In other words, the UPO for initial state good is to stay at the action pair (bad, good) throughout. Similarly, the UPO for initial state bad is to stay at the action pair (good, bad) throughout. Note that in the UPO, the pair of actions, (bad, bad) or (good, good), yielding (0, 0) are never played.

Finite Horizon SPE

The following result is proved in Section 7. It shows that the equilibrium involves the sub-optimal play of (good, good):

**Proposition 6** There is a unique SPE and the associated strategies are to play good regardless of history when there are an even number of periods left and to play the myopic best response - bad against good, and good against bad - when there are an odd number of periods left. Hence, if the initial state is good and \( T \) is even or if the initial state is bad and \( T \) is odd, then within the first two periods the mover does not play a myopic best response, i.e., period payoffs are (0, 0). Hence, SPE play is in those cases sub-optimal.

4.2.3 Alternating Moves with Transfers

Clearly, the UPO solution is unchanged. We compute the transfers required to implement that solution and the associated payoffs.

\( \tau = 1 \) Regardless of the fixed previous period’s action, the UPO strategy picks the myopic best response. Hence

\[
\theta_1(b|g) = \theta_1(g|b) = 0.
\]

\( \tau = 2 \) Suppose that the fixed action is good, then \( \theta_2(b|g) \) is derived from

\[
2d + \theta_2(b|g) = 0 + c,
\]

that is,

\[
\theta_2(b|g) = c - 2d.
\]

If the fixed action is bad, then inducing the mover to play the UPO choice of \( g \) does not require transfers, that is, \( \theta_2(g|b) = 0 \).

\( \tau = 3 \) When the fixed action is good, \( \theta_3(b|g) \) is derived from the equation

\[
3d + \theta_3(b|g) = 0 + 2c - \theta_2(b|g),
\]

that is

\[
\theta_3(b|g) = c - d
\]

As before, if the fixed action is bad, it is \( \theta_3(g|b) = 0 \).

\( \tau = 4 \) The argument of \( \tau = 3 \) repeats. Indeed, when the fixed action is good, \( \theta_4(b|g) \) is derived from the equation

\[
4d + \theta_4(b|g) + \theta_2(b|g) = 0 + 3c - \theta_3(b|g),
\]

that is

\[
\theta_4(b|g) = c - d
\]

Once again, when bad is the fixed action, inducing the mover to select good does not require transfers, that is, \( \theta_4(g|b) = 0 \).

All this leads to
Proposition 7 There is a unique SPE; the associated strategies are to always play a myopic best response. The transfers are: at state good, the non-mover pays

\[ \theta_\tau(b|g) = \begin{cases} c - d & \text{if } \tau > 2 \\ c - 2d & \text{if } \tau = 2 \\ 0 & \tau = 1 \end{cases} \]

No transfers are paid at state bad, that is, \( \theta_\tau(g|b) = 0 \), all \( \tau \). Consequently, if \( T \) is even, we have \( V_T(b) = c - \frac{T-2}{2T}(c - d) \), \( \frac{W_T(b)}{T} = (c + d) - V_T(b) \), while

\[ \frac{V_T(g)}{T} = d + \frac{(T-2)}{2T}(c - d) - \frac{d}{T}, \quad \frac{W_T(g)}{T} = (c + d) - V_T(g) \].

Again, we finish this sub-section by providing a numerical example. If \( c = 3, d = 1 \), then a transfer of 2 is required every other period from the player whose preferred NE is getting played (to keep play at her "corner"). In this example, unlike the Prisoners dilemma, there is no trading of transfers. Only one player pays because she needs to "pay off" the other player.

Notice that payoffs for each player converge to \( \frac{1}{2}(c + d) \). So although the actions played are those that Player 1 may prefer, she pays Player 2 exactly that amount that equates their (Pareto-optimal) payoffs. It is a symmetric game, the action vector played is asymmetric but transfers restore payoff symmetry.

5 Two Generalizations

5.1 The N player Game

The reader might wonder if the result extends to the case of \( N \) players - and it does. Showing that is the point of this subsection. To keep notation to a minimum, we analyze the three player game. The proof will be seen to extend straightforwardly to any number \( N \geq 3 \) of players.

Let \( G \) now denote a three player stage game (in strategic form) with strategy sets \( A_i, A \equiv A_1 \times A_2 \times A_3 \) and payoff functions \( \pi_i : A \rightarrow \mathbb{R}, i = 1, 2, 3. \) Let \( i, j, k \) denote the generic players and suppose, without loss of generality, that the sequence of action choices is \( i \) followed by \( j \) followed by \( k \) - and then repeated.

5.1.1 Timing of Actions and Transfers

When there are three players, we have to be a bit careful about the timing of transfer commitments. In the two player game, in the period before player \( i \) moves, player \( j \) picks an action \( a_j \) and a transfer \( \theta_j \). From a strategic point of view, it is immaterial whether \( a_j \) and \( \theta_j \) are simultaneously chosen or the choice \( \theta_j \) follows the choice \( a_j \) (since the decision-maker is the same player).

Now, however, there are three decisions taken in the period prior; \( a_k \) along with two transfers \( \theta_{ji} \) and \( \theta_{ki} \) (both meant for player \( i \)). It is natural to
assume that the transfer commitments follow the choice $a_k$. We further adopt the convention that the order of transfer commitments follows the order in which actions were chosen; since $a_j$ precedes $a_k$, hence the choice $\theta_{ji}$ is assumed to precede $\theta_{ki}$. Put another way, in the period prior to $i$'s move, three choices are made in the following sequence: $a_j$, followed by $\theta_{ji}$, followed by $\theta_{ki}$.

If, at time $t$, it is player $i$'s turn to move, she can choose an action $a_{it}$ from $A_i$ - and this action then remains fixed till $t+3$. Denoting by $\bar{a}_{-it} = (a_{jt-2}, a_{kt-1})$ the (fixed) actions of the other players, the period $t$ stage game payoffs are $\pi_i(a_i), \pi_j(a_t), \pi_k(a_t)$, where $a_t = a_{it}, \bar{a}_{-it}$. Hence, the payoffs at $t$ of player $i$ inclusive of transfers are

$$\pi_i(a_i) + \theta_{ji,t-1}(a_{it}) + \theta_{ki,t-1}(a_{it})$$

while those of players $j$ and $k$ are

$$\pi_j(a_t) - \theta_{ji,t-1}(a_{it}), \pi_k(a_t) - \theta_{ki,t-1}(a_{it})$$

As before, the horizon $T$ is finite; lifetime payoffs are evaluated according to the undiscounted sum. If $k$ is the initial period mover, then there is an initial action state $\bar{a}_t, a_j$ for the game, which can be arbitrary. As in the two-player model, the initial transfer state $\theta_{ik}, \theta_{jk}$ is chosen by the non-movers in the "pre-period" - 1 (in the sequence $\theta_{ik}$ followed by $\theta_{jk}$).

### 5.1.2 Strategies and Equilibrium

As before, all stage game actions and past transfer commitments are observable and $h_t = (h_t^a, h_t^f)$ is the action-transfer history at date $t$. The action mover, player $i$, makes an action choice $a_{it}$ conditional on $h_t$. Thereafter, two transfer commitments - to next period’s mover $j$ - are made in the sequence $\theta_{ji,t}$ and $\theta_{ij,t}$ conditional, respectively, on $(h_t, a_{it})$ and $(h_t, a_{it}, \theta_{ki,t})$. A strategy vector defines a (possibly probabilistic) history $h_t$. Given strategies $\sigma = (\sigma_i, \sigma_j, \sigma_k)$, and for given history $h_t$, we denote by $h_{t+n}^a(a_{it}, \theta_{it})$ the history of length $t + n$ generated by $\sigma$ following $h_t$. Denote the lifetime payoff of the mover at time $t$, with, say $\tau = T - t$ periods left in the game, $V_{t\tau}$:

$$V_{t\tau}(h_t) = \max_{a_{it}} \{ [\pi_i(a_i, a_{-it}) + \theta_{ki,t-1}(a_{i}) + \theta_{ji,t-1}(a_{i})] + R_{t\tau-1}(h_t, a_{it}) \},$$

where $R_{t\tau-1}$ is

$$R_{t\tau-1}(h_t, a_{it}) = \max_{a_{jt}} \{ [\pi_i(a_j(h^*_{t+1}), a_{-j,t+1}) - \theta_{ij}(a_j(h^*_{t+1}))] + W_{t\tau-2}(h^*_{t+2}) \}$$

---

21Asynchronicity is essential for our results so the transfers cannot be chosen simultaneously by $j$ and $k$. We believe that the results would be unchanged though if the transfer sequence was reversed - to one in which $\theta_j$ precedes $\theta_k$.

22Note that a player picks an action $a_j$ every third period but she picks a transfer two out of every three periods, i.e., in every period that she is not the mover. In that precise sense, the transfer commitments are still one-period ahead commitments, as they are in the two player case. There is a variant in which a player picks (actions and) transfers every third period by putting down two transfer schedules - one for $t+1$ and another for $t+2$. This model has a similar on-equilibrium behavior as the one we report.
and \( W_{i\tau-2} \) is

\[
W_{i\tau-2}(h_t^{\tau+2}) = \max_{\theta_{ik}} \{ \pi_i(a_k(h_t^{\tau+2}), a_{-k,t+2}) - \theta_{ik}(a_k(h_t^{\tau+2})) \} + V_{i\tau-3}(h_t^{\tau+3}) \}.
\]

The **Utilitarian Pareto Optimum** (UPO) problem is the maximization of the lifetime sum of players payoffs

\[
\max_{\{a_{it}, a_{jt}, a_{kt}\}_{t \geq 0}} \sum_{t=0}^{T} \left[ (\pi_i + \pi_j + \pi_k)(a_{it}, a_{jt}, a_{kt}) \right]
\]

subject to an initial action state \((\bar{a}_i, \bar{a}_j)\) and that player \(i\) can change her action in periods 1, 4, ... while player \(j\) can do so in periods 2, 5, ... and, finally, player \(k\) moves in periods 0, 3, ...

It is well known that the solution is Markovian, i.e., the Pareto optimal solution is for player \(i\) to play according to \(\pi_i\) whenever it is her turn to move and \(\pi\) periods remain.

5.1.3 **A General Result for the Finite Horizon**

The main theorem that ties equilibrium behavior to the UPO is:

**Theorem 8** There is a unique SPE to the game. It coincides with the Pareto optimum solution in terms of actions - with \(\tau\) periods to go, the equilibrium is for player \(i\) to play according to \(\pi_i\) whenever it is her turn to move.

The proof is in Section 6 but the principle of the construction is worth a brief explanation. In the two player game, recall, the player picking the transfer (in the immediately preceding period) acted like a Stackleberg leader. She picks the transfer to maximize her net payoff subject to the subsequent period’s mover picking a best response action for every transfer chosen. In particular, she holds the subsequent mover to his best payoff absent transfers, i.e., holds him to a constant irrespective of what action he chooses. Thereby, she has an incentive to maximize the sum of their payoffs since her net is maximized if and only if the sum is maximized.

In the three player game, there are two players picking transfers in the previous period in sequence. The first of them acts like the Stackleberg leader. **She holds the subsequent two movers to their best payoff absent transfers**, i.e., she holds them to a constant irrespective of what transfer and action they subsequently choose. Again, her choice of transfer then maximizes the (three player) sum of payoffs. The second player picking a transfer, typically but not always, acts as the residual transferee getting the desired utilitarian action taken by paying the balance of the aggregate transfer.
### 5.2 General Asynchronous Games

A question that the discerning reader will have is whether the order of moves studied in this paper—players alternating their move and hence each player moving exactly once in between two consecutive moves of her opponent—is important for the results. The answer is no. Indeed consider the following general definition:

**Definition 9** An *asynchronous game* (with horizon $T$ and stage game $G$) is described by a player assignment function $X : \{0, 1, \ldots, T\} \rightarrow \{1, 2, \ldots, N\}$. At time $t$, player $X(t)$ - and only that player - gets to move.

From the assignment function, $X$, one can deduce next move assignment functions $\Phi_i$, one for each player, that specify for every period $t$ and every player $i$ a next time of move for that player, $\Phi_i(t)$. Consider the following straightforward extension of our $N$ player transfer game:

**Definition 10** An *asynchronous game with transfers* is described by $\Phi_i$ and the following transfer possibilities: Whenever $\Phi_i(t) = t + 1$, there are no transfer possibilities (from player $i$ to herself). However, if $\Phi_i(t) > t + 1$, then at $\Phi_i(t) - 1$, players other than $i$ can make a one-period ahead transfer commitment to player $i$, $\theta_{ji, \Phi_i(t)}$, $j \neq i$. These commitments are made sequentially in the same order in which the actions in $\bar{a}_{-i}$ were played, where $\bar{a}_{-i}$ is the fixed action vector faced by $k$, the mover in period $\Phi_i(t) - 1$.

It is straightforward to define the Utilitarian Solution in this setting. It is also straightforward to see that our main theorem applies here—there is a unique SPE that corresponds with the UPO.

**Theorem 11** There is a unique SPE to the game. It coincides with the Pareto optimum solution in terms of actions—with $\tau$ periods to go, the equilibrium is for player $i$ to play according to $\tilde{\beta}_i(\bar{a}_{-i, \tau}; \tau)$ whenever it is her turn to move.

The proof follows the same logic of the argument for Theorem 10 and it is therefore omitted.

### 5.3 Transfers with No Commitment

It is critical that a player can commit for one period to a chosen transfer schedule. Evidently if she cannot, then in the very last period, she will renege. Knowing this the last period’s mover will always choose a myopic best response. In turn that implies that the penultimate period’s transfer schedule will be reneged on as well. Evidently, then the SPE of such a game will merely coincide with the SPE of the asynchronous game without transfers. We will (generically) get unique SPE but they will, typically, not be efficient.
6 Conclusion

In this paper we show that in finite horizon asynchronous move games, in which players can make one-period ahead transfer commitments, there is a unique equilibrium and it coincides with the Utilitarian Solution. Since the latter is a solution to a Dynamic Programming problem, it is easily computed (and is a far simpler computational problem than a fixed point problem). Finally, the required equilibrium transfers are easy to compute as well.

This paper is a start towards introducing Mechanism Design in Dynamic Games. We do not have to be trapped in the "anything goes" of Folk Theorems. Simple modifications in game form - which a Mechanism Designer can make - produce dramatic changes in equilibrium outcomes.

7 Appendix

7.1 Computations of Section 4

Battle of the Sexes

Proof of Proposition 6: Without loss of generality, suppose that the mover in the last period is Player 1.

\(\tau = 1\) Since bad is the best response to good, and good is the best response to bad, bad is the best response to good and good is the best response to good. Hence, \(v_1(g) = d, w_1(g) = c\) while \(v_1(b) = c, w_1(b) = d\).

\(\tau = 2\) Suppose good was the previous period’s action. Then by playing g, the mover gets \(0 + w_1(g) = c\) while by playing b the mover gets \(d + w_1(b) = 2d\). Clearly g is the better move. Hence, \(v_2(g) = c, w_2(g) = d\). Equally clearly, g is the best response if bad was the previous period’s action since it yields a higher current and continuation payoff, i.e., \(v_2(b) = 2c, w_2(b) = 2d\).

\(\tau = 3\) Suppose good was the previous period’s action. Then by playing g, the mover gets \(0 + w_2(g) = d\) while by playing b the mover gets \(d + w_2(b) = 3d\). Now b is the better move since it yields higher current and continuation payoffs. Hence, \(v_3(g) = 3d, w_3(g) = 3c\). If bad was the fixed action, then by playing g, the mover gets \(c + w_2(g) = c + d\) while by playing b the mover gets \(0 + w_2(b) = 2d\) and hence g is the better move. Hence, \(v_3(b) = c + d, w_3(b) = c + d\).

\(\tau = 4\) This looks like \(\tau = 2\) since \(w_3(g) - w_3(b) = 3(c - d)\) thereby overwhelming the immediate period’s payoff difference. Hence, g becomes the best response regardless of the fixed action from the previous period.

Clearly the argument of \(\tau = 4\) repeats to any period \(4 + 2n\), while the argument of \(\tau = 3\) repeats to any period \(3 + 2n\), for \(n = 1, 2, \ldots\). Hence we have proved the proposition. ■
7.2 Proofs of Section 5

7.2.1 Proof of Theorem 10

One Remaining Period $\tau = 1$ As in the two player game, we proceed by backward induction. Suppose that $k$ is the last player to move; denote the choice $a_k$. That means two transfers were chosen in the prior period $\tau = 1$; $\theta_{ik,1}$ - the transfer to player $k$ from player $i$ - followed by $\theta_{jk,1}$. When defining a maximization problem, we put a bar on predetermined variables; for example, $a_k$ is determined given $\bar{\theta}_{ik,1}, \theta_{jk,1}$.

Determination of the equilibrium map $(\theta_{ik,1}, \theta_{jk,1}, a_k)$:
Player $k$ solves:

$$\arg \max_{a_k} \pi_k(a_k, \bar{a}_{-k}) + \bar{\theta}_{ik,1}(a_k) + \bar{\theta}_{jk,1}(a_k)$$ (12)

Denote the value function of problem 12 $V_{k0}(\cdot)$ and $a_k(\cdot)$ is the maximizer.\footnote{Note that values and choices are only going to depend on payoff relevant history. For example, $a_k$ will only depend on the immediately prior transfers and the two immediately prior actions but not on actions and transfers prior to those periods. Second, in order to save on notation, whenever there is no possibility of confusion, we do not make explicit the dependence on history.}

The transfer choice of Player $j$, $\theta_{jk,1}$, takes as given $\bar{\theta}_{ik,1}, \bar{a}_{-k}$ and, from above, the function $a_k(\cdot)$:

$$\max_{a_k, \bar{\theta}_{jk,1}} \pi_j(a_k, \bar{a}_{-k}) - \theta_{jk,1}(a_k)$$ (13)

$$s.t. a_k \text{ is a solution to } 12.$$

By notation, $R_{j0}$ is the value function of problem 13. Finally, the transfer choice of Player $i$, $\theta_{ik,1}$ solves:

$$\max_{a_k, \bar{\theta}_{ik,1}, \bar{\theta}_{jk,1}} \pi_i(a_k, \bar{a}_{-k}) - \theta_{ik,1}(a_k)$$ (14)

$$s.t. a_k, \bar{\theta}_{jk,1} \text{ is a solution to } 13.$$

By notation, $W_{i0}$ is the value function of problem 14

The next lemma characterizes the SPE choice of $\theta_{jk,1}$ and $a_k$, given $\bar{\theta}_{ik,1}$; denote them $a^\ast_k$ and $\bar{\theta}^\ast_{jk,1}$. Note that once player $i$ has determined the transfer schedule $\bar{\theta}_{ik,1}$, then players $j$ and $k$ are in a position that is exactly a replica of the two player scenario we have already studied. Player $k$, the follower, picks his most preferred action and player $j$, the leader hence maximizes the "surplus" $\pi_k(a_k, \bar{a}_{-k}) + \pi_j(a_k, \bar{a}_{-k}) + \bar{\theta}_{ik,1}(a_k)$ and uses that maximization to "allocate" an action to player $k$. We note that in the lemma below but since the proof is already known from the previous section, we omit it here.

Lemma 12 Let $\bar{\theta}_{ik,1}, \bar{a}_{-k}$ be given. Then, the equilibrium action of player $k$ and the transfers from player $i$ to $k$ are given by

$$a^\ast_k \in \arg \max_{a_k} \{\pi_k(a_k, \bar{a}_{-k}) + \pi_j(a_k, \bar{a}_{-k}) + \bar{\theta}_{ik,1}(a_k)\}$$
Corollary 13
Let \( \theta^e_{ik,1}(a_k) \) be a solution to 14. Then,
\[
\theta^e_{jk,1}(a_k|\theta_{ik,1}) = \begin{cases} 
0, & \text{if } a_k \neq a^e_k \\
\max_{a_k} \{ \pi_k(a_k, a_{-k}) + \theta_{ik,1}(a_k) \} - \{ \pi_k(a_k, a_{-k}) + \theta_{ik,1}(a_k) \}, & \text{if } a_k = a^e_k 
\end{cases}
\]

and equilibrium payoffs are
\[
\max_{a_k} \{ \pi_k(a_k, a_{-k}) + \theta_{ik,1}(a_k) \}, \text{ for player } k
\]
\[
\max_{a_k} \{ \pi_k(a_k, a_{-k}) + \theta_{ik,1}(a_k) \} - \max_{a_k} \{ \pi_k(a_k, a_{-k}) + \theta_{ik,1}(a_k) \}, \text{ for player } j
\]

Lemma 12 provides bounds on the equilibrium payoffs of the players. These bounds will be useful in characterizing the equilibrium values of \((a_k, \theta_{ik,1}, \theta_{jk,1})\).

Proof: The first inequality follows immediately from the definition of players’ payoffs. The second and third inequality follow directly from Lemma 12 by taking into account that \( \theta^e_{ik,1} \) is non-negative.

The next Lemma characterizes the SPE choice of \( \theta_{ik,1} \):

Lemma 14
Let \((a^e_k, \theta^e_{ik,1}, \theta^e_{jk,1})\) be a solution to 14. Then, \( a^e_k \) is the utilitarian Pareto optimal action
\[
a^e_k \in \arg \max_{a_k} (\pi_1 + \pi_2 + \pi_3)(a_k, a_{-k}),
\]

the equilibrium transfer from player \( j \) to \( k \) is given by
\[
\theta^e_{jk,1}(a_k) = \begin{cases} 
0, & \text{if } a_k \neq a^e_k \\
\max \{ 0, \pi_j(a_k, a_{-k}) - R^*_j \}, & \text{if } a_k = a^e_k 
\end{cases}
\]
\[ (15) \]
equilibrium transfers from player 1 to 3 are
\[
\theta^e_{ik,1}(a_k) = \begin{cases} 
V^*_k + R^*_j - (\pi_j + \pi_k)(a^e_k, a_{-k}), & \text{if } a_k = a^e_k 
\end{cases}
\]
\[ (16) \]
and equilibrium payoffs are
\[
V^*_k = V^*_k + \max \{ 0, R^*_j - \pi_j(a^e_k, a_{-k}) \},
\]
\[
R^*_j = \min \{ \pi_j(a^e_k, a_{-k}), R^*_j \},
\]
\[
W^*_1 = U_1 - \{ V^*_k + R^*_j \}
\]
Proof: First we show that, given \( \theta_{jk,1}, \theta_{ik,1} \), the action \( a_k^e \) is a solution to problem 12. After substitution, the payoffs of player \( k \) become

\[
\pi_k(a_k, \bar{a}_{-k}), \text{ for } a_k \neq a_k^e,
\]

while for \( a_k = a_k^e \), they are

\[
V_{k1}^{*}, \text{ if } \pi_j(a_k^e, \bar{a}_{-k}) \geq R_{j1}^*\]

and

\[
V_{k1}^{*} + R_{j1}^* - \pi_j(a_k^e, \bar{a}_{-k}), \text{ if } \pi_j(a_k^e, \bar{a}_{-k}) < R_{j1}^*.
\]

Thus, the payoff at \( a_k^e \) is never less than \( V_{k1}^{*} \), while by definition, the payoff at \( a_k \neq a_k^e \) is never more than \( V_{k1}^{*} \). Thus, \( a_k^e \) is an optimal action choice for player \( k \).

Second we show that given \( \theta_{ik,1} \), the action-transfer pair \( a_k^e, \theta_{jk,1} \) is a solution to problem 13. By Lemma 12, this simplifies to showing that \( a_k^e \) maximizes \((\pi_j + \pi_k)(a_k, \bar{a}_{-k}) + \theta_{ik,1}(a_k)\), that is, substituting for the expression of \( \theta_{ik,1}(a_k) \), it maximizes the function:

\[
(\pi_j + \pi_k)(a_k, \bar{a}_{-k}), \text{ if } a_k \neq a_k^e
\]

and

\[
V_{k1}^{*} + R_{j1}^*, \text{ if } a_k = a_k^e.
\]

The definition of \( V_{k1}^{*} + R_{j1}^* \) implies the claim.

Before showing that \( a_k^e, \theta_{jk,1} \) is a solution to problem 14, notice that by direct computation \( W_{i1}, V_{k1}, R_{j1} \) are the payoffs obtained substituting for the expressions of \( (a_k^e, \theta_{ik,1}, \theta_{jk,1}) \) in the players’ payoff functions. The optimality of \( a_k^e, \theta_{jk,1} \) follows from the Corollary. Indeed, by the Corollary, player \( i \)'s equilibrium payoff cannot exceed \( U_1 - \{V_{k1}^{*} + R_{j1}^*\} \). Therefore, the equality \( W_{i1} = U_1 - \{V_{k1}^{*} + R_{j1}^*\} \), verified at equilibrium, implies the claim. □

Two Remaining Periods \( \tau = 2 \) Determination of the equilibrium maps \( (\theta_{kj,2}, \theta_{ij,2}, a_j) \)

Now player \( j \) is the mover and solves:

\[
\arg \max_{a_j} \pi_j(a_j, \bar{a}_{-j}) + \bar{\theta}_{ij,2}(a_j) + \bar{\theta}_{kj,2}(a_j) + R_{j1}(a_j) \quad (17)
\]

By notation, the value function of problem 17 is \( V_{j2} \) and \( a_j(\cdot) \) the maximizer. Taking that as given, player \( i \) solves

\[
\max_{a_j, \theta_{ij,2}} \pi_i(a_j, \bar{a}_{-j}) - \theta_{ij,2}(a_j) + W_{i1}(a_j) \quad (18)
\]

s.t. \( a_j \) is a solution to 17.

By notation, \( R_{i2} \) is the value function of problem 18.
Finally, since player $k$ picks his transfer first, he solves:

$$
\max_{a_j, \theta_{ij,2}, \theta_{kj,2}} \pi_k(a_j, \bar{a}_{-j}) - \theta_{kj,2}(a_j) + V_{k1}(a_j) \\
\text{s.t. } a_j, \theta_{ij,2} \text{ is a solution to } 18.
$$

By notation, $W_{i2}$ is the value function of problem 19.

Now we state the equivalent of results we saw for $\tau = 1$. The equivalent of Lemma 12 is:

**Lemma 15** Let $(\bar{a}_{-j}, \bar{\theta}_{kj,2})$ be given. Then the equilibrium action chosen by player $j$, $a_j^*$, is the solution to

$$
\max_{a_j} (\pi_i + \pi_j)(a_j, \bar{a}_{-j}) + (\theta_{kj,2} + R_{j1} + W_{i1})(a_j),
$$

while the equilibrium transfer from player $i$ to $j$ is

$$
\theta_{ij,2}(a_j|\bar{\theta}_{kj,2}) = \begin{cases} 
0, & \text{if } a_j \neq a_j^* \\
\max_{a_j} [\pi_j(a_j, \bar{a}_{-j}) + (\theta_{kj,2} + R_{j1})(a_j)] - \pi_j(a_j, \bar{a}_{-j}), & \text{if } a_j = a_j^* 
\end{cases}
$$

and equilibrium payoffs are

$$
\max_{a_j} [\pi_j(a_j, \bar{a}_{-j}) + (\theta_{kj,2} + R_{j1})(a_j)], \text{ player } j,
$$

while

$$
\max_{a_j} [\pi_j(a_j, \bar{a}_{-j}) + (\theta_{kj,2} + R_{j1})(a_j)] \\
- \max_{a_j} [\pi_j(a_j, \bar{a}_{-j}) + (\bar{\theta}_{kj,2} + R_{j1})(a_j)], \text{ player } i
$$

The equivalent of the Corollary takes the form:

**Corollary 16** The equilibrium payoffs satisfy

$$
W_{k2} + R_{i2} + V_{j2} \leq U_2 = \max_{a_j} (\pi_1 + \pi_2 + \pi_3)(a_j, \bar{a}_{-j}) + U_1(a_j)
$$

$$
R_{i2} + V_{j2} \geq R^*_{i2} + V^*_{j2} = \max_{a_j} (\pi_j + \pi_i)(a_j, \bar{a}_{-j}) + (R_{j1} + W_{i1})(a_j)
$$

and

$$
V_{j2} \geq V^*_{j2} = \max_{a_j} \pi_j(a_j, \bar{a}_{-j}) + R_{j1}(a_j)
$$

The equivalent of Lemma 14 is now:
Lemma 17 Let \((a_j^e, \theta_{ij}^e, \theta_{kj}^e)\) be MPE values, that is, a solution to 19. Then, \(a_j^e\) is the utilitarian Pareto optimal action
\[
a_j^e \in \arg \max_{a_j} (\pi_1 + \pi_2 + \pi_3)(a_j, \bar{a}_{-j}) + U_1(a_j)
\]
and equilibrium transfers from player \(i\) to \(j\) are
\[
\theta_{ij}^e(a_j) = \begin{cases} 
0, & \text{if } a_j \neq a_j^e \\
\max\{0, \pi_i(a_j, \bar{a}_{-j}) + V_{i1}(a_j) - R_{i2}^e\}, & \text{if } a_j = a_j^e
\end{cases}
\]
equilibrium transfers from player \(k\) to \(j\) are
\[
\theta_{kj}^e(a_j) = \begin{cases} 
0, & \text{if } a_j \neq a_j^e \\
V_{j2} + R_{i2}^e - \{(\pi_i + \pi_j)(a_j, \bar{a}_{-j}) + (V_{i1} + R_{j1})(a_j)\}, & \text{if } a_j = a_j^e
\end{cases}
\]
and equilibrium payoffs are
\[
\begin{align*}
V_{2j} &= V_{j2}^e + \max\{0, R_{i2}^e - \pi_i(a_j^e, \bar{a}_{-j}) - V_{i1}(a_j^e)\}, \\
R_{i2} &= \min\{\pi_i(a_j^e, \bar{a}_{-j}) + V_{i1}(a_j^e), R_{i2}^e\}, \\
W_{k2} &= U_2 - \{V_{j2}^e + R_{i2}^e\}
\end{align*}
\]
Arguing inductively, for periods \(\tau\) when player \(i\) is the mover, the equilibrium action, \(a_{i\tau}^e\), is the maximizer of the Utilitarian problem, and \(U_{\tau-1}\) is the value of the Utilitarian problem
\[
U_\tau = \max_{a_{i\tau}} (\pi_1 + \pi_2 + \pi_3)(a_{i\tau}, a_{-i\tau}) + U_{\tau-1}(a_{i\tau})
\]
while equilibrium transfers conform to the expressions given in Lemma 17 with the appropriate changes of indices. Thus, the same proof as above applies to give us the result.

References


