Advancements in Gravity Models of Spatial Economics

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A Unified Framework for Spatial Economics

- Many sciences, as well as disciplines in economics, based on unified setup

- Urban: Rosen-Roback spatial equilibrium model (Glaeser Gottlieb '09)

- This model lacks spatial linkages/frictions

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- I argue that this new setup has large advantages

- Offers comprehensive analytical framework for spatial economics...
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    - Offers comprehensive analytical framework for spatial economics...
    - ...integrated framework for fields of trade, geography, and urban
A Unified Framework for Spatial Econ: What do we Need?

- This framework ought to satisfy the following properties
  1. Analytically tractable but rich
     - Micro-foundations
     - Analytical expressions from consumer/firm choice and nice aggregation
     - Flexible enough to model complicated spatial linkages
  2. Have a clear mapping to the data
     - Model variables correspond to national statistics & link to micro data
     - Offer an easy setup to estimate key parameters
  3. Have desirable theoretical properties
     - Positive (existence, uniqueness, comparative statics)
     - Normative (link welfare and openness)
     - Easy to work with/compute
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A Unified Framework for Spatial Econ: The Gravity Model

- General Equilibrium gravity model easily passes first two tests
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     - The gravity framework is a daunting black box
     - Its empirical success notwithstanding, until recently, little could be said about its properties
What About the Theoretical Properties of the Model?

- The GE gravity model is a hard model to solve!

- In the best case scenario, N country equations/unknowns + GE interactions
- Makes for a problem with a very formidable solution
- Problem in economic geography (i.e. when labor is mobile) urban (e.g. knowledge spillovers) can be a true nightmare: agglomeration externalities
- Work in the past 5 years offered sharp characterization of gravity model:
  - Extremely versatile setup. Works well for trade, geography, urban econ
- A volley of mathematical tools can be used to characterize its theoretical properties, e.g. non-linear equations theory, integral equations etc.
- Discussion based on a rapidly expanding literature:
  - Will be discussed using results/model in Allen Arkolakis (AA) '14, AA Takahashi '14 (AAT), AA and Li '14 (AAL14), AA and Li '15 (AAL15) and some earlier results by Arkolakis Costinot Rodriguez-Clare (ACR) '12
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Roadmap

- Analytical Gravity and Mapping to the Data
- Gravity, Modules, and Models
- Characterization of Urban Equilibrium
- Applications
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Trade and Commuting Gravity: Setup

- Gravity trade model (Anderson ’79):
  - perfect competition, each location produces a differentiated variety
  - CES preferences with elasticity $\sigma$ across varieties.

- Bilateral trade given by

$$X_{ij} = \frac{\left(\frac{w_i}{A_i} \tau_{ij}\right)^{1-\sigma}}{\sum_k \left(\frac{w_k}{A_k} \tau_{kj}\right)^{1-\sigma} E_j}$$

  - $A_i$ is productivity, $\tau_{ij}$ is iceberg cost, $w_i$, is wage rate, $E_j$, country spending
  - We call $\epsilon \equiv 1 - \sigma$ the ‘trade elasticity’
Trade Gravity: Analytics and Micro-foundations

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- Other microfoundations

  - EK offered microfoundations for DFS: Frechet distributed productivities
    - Bergstrand '85 in partial equilibrium, Arkolakis Klenow Demidova Rodriguez-Clare '08 in GE, gravity for Krugman '80
    - Chaney '08 for monopolistic competition with heter. firms (Melitz '03)
    - Bernard, Eaton, Jensen and Kortum '03 gravity with Bertrand competition
Use formulation of AAL15 ($L_i$: workers, $\mu_{ij}$: commuting cost, $u_i$: amenities)

- Step 1: Choose work location $\arg\max W_j$. Spatial equilibrium: $W_j = W$.
- Step 2: Choose living location after observing Frechet preference shock as in Ahlfedlt et al: $F(v) = e^{-v^{-\theta}}$. 

In expectation welfare of working in $j$ is $W_j \equiv E\left[\max_i (u_i w_j P_i \mu_{ij} \nu_i(\omega))\right] = w_j \left(\sum_i (u_i P_i \mu_{ij})\right)^{\frac{1}{\theta}}$. 

From step 1 we get bilateral commuting flows $L_{ij} = (u_i P_i \mu_{ij}) \theta \sum_k (u_k P_k \mu_{kj}) \theta L_j = W_j - \theta (\mu_{ij}) - \theta (u_i P_i) \theta w_j L_j$. 

Urban Gravity: Analytics and Micro-foundations
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▶ From step 1 we get bilateral commuting flows

$$L_{ij} = \frac{\left( \frac{u_i}{P_i \mu_{ij}} \right)^\theta}{\sum_k \left( \frac{u_k}{P_k \mu_{kj}} \right)^\theta} L_j = W^{-\theta} (\mu_{ij})^{-\theta} \left( \frac{u_i}{P_i} \right)^\theta w_j^\theta L_j.$$
Gravity Model and the Aggregate Data

- The quintessential example of an applied framework
  - Empirical counterpart for aggregate variables (GDP trade prices tariffs)

- The Idea (focus on trade): We can write bilateral trade flows, $X_{ij}$, as

$$X_{ij} = \gamma_i \delta_j \tau_{ij}^\epsilon \implies \ln X_{ij} = \epsilon \ln \tau_{ij} + \ln \gamma_i + \ln \delta_j$$

- Early formulation: reduced form gravity relationship e.g. Tinberger '62
- Can formulate fixed effects specification (EK/Redding Venables)
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- Independent of microfoundations, aggregate parameters (e.g. $\epsilon$) estimated with direct measures of trade costs (e.g. tariffs):
  - See Caliendo Parro '15, Arkolakis, Ramondo, Rodriguez-Clare, Yeaple '13
  - Exploit micro structure: EK, Donaldson '14, Simonovska Waugh '14 '15

- Other approaches involving using the GE structure for estimation
  - Anderson van Wincoop '04, Bergstrand, Egger and Larch, AAT
Gravity Model and the Micro Data

- The quintessential example of an applied framework
  - Micro-data can be used without affecting macro structure
  - As in Bernard Eaton Jensen Kortum '03, EK Kramarz '11, Arkolakis '10
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- This model has been recently adapted to
  - economic geography (AA, Redding '15, Ramondo et al, Caliendo et al)
  - urban (Ahlfedlt, Redding, Sturm, Wolf '15, AAL15, Monte et al '15)
Roadmap

- Analytical Gravity and Mapping to the Data
- Gravity, Modules, and Models
- Characterization of Equilibrium
- Applications
General Equilibrium

- We discussed how to create spatial linkages across locations (e.g. cities, countries) using gravity equations
  - Trade or commuting are just two examples

- Next step: close model to compute equilibrium, welfare, comparative statics
  - Goal: formulate a computable system of equations/unknowns
A GE model with trade needs to satisfy two accounting conditions:

- “Goods market clearing”:
  \[ Y_i = \sum_{j \in S} X_{ij} \quad \forall i \in S \]

- “Budget Balance”:
  \[ E_i = \sum_{j \in S} X_{ji} \quad \forall i \in S \]

Note: it may be that \( E_i \neq Y_i \)
Accounting: Urban Module

- A GE model with urban flows (commuting) needs to satisfy accounting as well
  - Total output in $i$ is equal to total earnings:
    \[ Y_i = \sum_j w_i L_{ji} \]  
    (1)
  - Total spending in $i$ is equal to what earned everywhere:
    \[ E_i = \sum_j w_j L_{ij} \]  
    (2)

- We developed essential components for trade, geography, urban models
  - Gravity (trade and commuting)
  - Accounting modules (trade and urban)
- Now we define and analyze the GE of these models
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Closing the Trade Model

- In trade models (with no deficit) we have $E_i = Y_i$
  - Labor is the only factor so $Y_i = w_i L_i$

- Equilibrium is trade gravity + trade module.
  - Solve $w_i, P_i$ using

$$w_i^\sigma = \sum_{j \in S} (\tau_{is})^{1-\sigma} L_i^{-1} A_i^{\sigma-1} \tilde{u}_j^{\sigma-1} L_j w_j P_j^{\sigma-1}$$

$$P_i^{1-\sigma} = \sum_{j \in S} (\tau_{ji})^{1-\sigma} A_j^{\sigma-1} (w_j)^{1-\sigma}$$
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$$P_i^{1-\sigma} = \sum_{j \in S} (\tau_{ji})^{1-\sigma} A_j^{\sigma-1} (w_j)^{1-\sigma}$$

- We intentionally avoided substituting the price index.
  - Crucial to write it this way, as it is much easier to characterize
  - AAT show that equilibrium exists and is unique if $\sigma > 1$
  - Restricting $\tau_{kj} = \tau_{jk}$ implies a milder restriction for uniqueness, $\sigma > \frac{1}{2}$
In economic geography, as in AA, we model local spillovers:

- production $A_i = \bar{A}_i L_i^{\alpha}$, amenity $u_i = \bar{u}_i L_i^{\beta}$
In economic geography, as in AA, we model local spillovers:

- production $A_i = \bar{A}_i L_i^{\tilde{\alpha}}$,
- amenity $u_i = \bar{u}_i L_i^{\tilde{\beta}}$

Different $\tilde{\alpha}$, $\tilde{\beta}$ isomorphic to different economic geography models

- E.g. Monopolistic competition with free entry: $\tilde{\alpha} = \frac{1}{\sigma - 1}$.
- Cobb-Douglas preferences over non-tradable sector: $\tilde{\beta} = - \frac{1-\gamma}{\gamma}$.
Geography Model: Equilibrium Equations

- Equilibrium is gravity+trade module+
  - Utility equalization $W_i = W$
  - Aggregate labor clears $\sum_i L_i = \bar{L}$

- Solve $w_i, L_i, W$ using

$$W^{\sigma-1} L_i^{1-\bar{\alpha}(\sigma-1)} w_i^\sigma = \sum_{j=1}^{N} T_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \bar{u}_j^{\sigma-1} L_j^{1+\bar{\beta}(\sigma-1)} w_s^\sigma$$

$$W^{\sigma-1} w_i^{1-\sigma} L_i^{\bar{\beta}(1-\sigma)} = \sum_{j=1}^{N} T_{ji}^{1-\sigma} \bar{A}_j^{\sigma-1} \bar{u}_i^{\sigma-1} w_j^{1-\sigma} L_j^{\bar{\alpha}(\sigma-1)}$$

and of course $\sum_i L_i = \bar{L}$.

- Existence and uniqueness in AA and AAT: notice same mathematical structure as in the trade model.
  - Except now welfare is the eigenvalue of the system
Summary of GE Gravity Trade & Geography Models

- GE gravity trade (Anderson '79: solve for $w_i, P_i$)

$$w_i^\sigma L_i = \sum_{j=1}^{N} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} L_j P_j^{\sigma-1} w_j$$

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\[
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\]

- GE geography (AA: welfare equalizes, solve for $W = \frac{w_i}{P_i}, w_i, L_i$)

\[
W^{\sigma-1} L_i^{1-\bar{\alpha}(\sigma-1)} w_i^\sigma = \sum_{j=1}^{N} T_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \bar{u}_j^{\sigma-1} L_j^{1+\bar{\beta}(\sigma-1)} w_j^\sigma
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W^{\sigma-1} w_i^{1-\sigma} L_i^{\bar{\beta}(1-\sigma)} = \sum_{j=1}^{N} T_{ji}^{1-\sigma} \bar{A}_j^{\sigma-1} \bar{u}_i^{\sigma-1} w_j^{1-\sigma} L_j^{\bar{\alpha}(\sigma-1)}
\]

- Now $W^{\sigma-1}$ is the eigenvalue (and total population constraint $\sum_j L_j = \bar{L}$)
Urban Model: Spatial Spillovers

- We now turn to consider the urban model

\[ A_i = \sum_j K_{ij} \left( L_j \right)^{\eta} \]

- \( K_{ij} \) represents spatial knowledge links.
- \( \eta \) is the degree of spillover.
- Microfoundations for this functional form presented in AAL '15
Urban Model: Spatial Spillovers

- We now turn to consider the urban model

- Agglomerations are important for cities’ economic activity (Fujita Thisse ’02, Glaeser Gottlieb ’09, Moretti ’11, Davis Dingel ’12)
  - A most crucial: spatial knowledge spillover (introduced by Fujita-Ogawa ’82)
  - Turns out: easy to extend this framework to introduce this spatial spillover
  - Assume that productivity in a location depend on the number of spatial interactions with other nearby workers, $L_j$

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- $K_{ij}$ represents spatial knowledge links. $\eta$ is the degree of spillover
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Closing the Urban Model

- Equilibrium is
  - gravity for trade+trade module+
  - gravity for commuting+urban module+
  - spatial spillovers

- Solve for $E_i$, $Y_i$, $L_i$, $w_i$, $A_i$, $W$, in the following 5 equations

  trade module: $Y_i = \sum_{j \in S} X_{ij}$, $E_i = \sum_{j \in S} X_{ji}$

  urban module: $E_i = \sum_j w_j L_{ij}$, $Y_i = \sum_j w_i L_{ji}$

  spatial spillover: $A_i = \sum_j K_{ij}(L_j)^\eta$

This general structure incorporates all the previous models as subcases

- Trade module: Armington, AA, AAT
- Urban module: Ahfelt, Redding, Sturm (two factors)
- Trade+urban+spatial spillovers module: AAL
Urban Models: Taking Stock

- That appears like a daunting system to deal with!
  - Turns out it is possible to make further progress
Urban Models: Taking Stock

- That appears like a daunting system to deal with!
  - Turns out it is possible to make further progress
  - Analytical solutions adapted from the theory of differential/integral equations (Fabinger ’15, AA, AAL15)
  - AAL ’14 generalize proof of existence & uniqueness from operator theory
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Fujita Ogawa Model with Gravity

- Fujita Ogawa introduced spatial knowledge/productivity spillovers in an urban model
  - Until now analysis focused on explicitly calculating model equilibria (Fujita Ogawa ’82, Lucas Rossi-Hansberg ’03, Rossi-Hansberg ’05)
  - Our gravity model can also be characterized in various special cases
Analytical Solution in the Circle

- We consider the circle
  - Assume commuting model in a circle $[-\pi, \pi]$, no trade costs
  - Spatial knowledge spillovers $K_{ij} = \cos^2\left(\frac{x-s}{2}\right)$, $x, s \in [-\pi, \pi]$

- Solution $L(x) = (C_1 + C_2 \sin(x + C_3))^{(\sigma-1)}$
  - $C_1 > C_2 \geq 0$ are determined by normalization $C_3$ can be arbitrary.

- If $|\eta(\sigma - 1)| \leq 1$ unique equilibrium if $\sigma > 1$ multiple (any point can be the center)
Labor distribution: $C_1 + C_2 \sin(x - \pi/2)$
Analytical Solution

Labor distribution: \( C_1 + C_2 \sin(x - \pi/2) \)

\( \eta = 0.8 \quad \sigma = 3 \)
Analytical Solution

Labor distribution: \( C_1 + C_2 \sin(x - \pi/2) \)

\[ \eta = 1 \quad \sigma = 3 \]
Analytical Solution

Labor distribution: $C_1 + C_2 \sin(x - \pi/2)$

$\eta = 1.2 \quad \sigma = 3$
Analytical Solution

Labor distribution: $C_1 + C_2 \sin(x - \pi/2)$

$\eta = 1.4 \quad \sigma = 3$
A Generalized Gravity ‘Model’

Suppose equilibrium of a model reduces to a system of \((H \times N)\) eqns where we denote locations (or sectors/location-sectors) with \(i, j \in \{1, \ldots, N\}\), eqns with \(k\), type of variable with \(h\); \(k, h \in \{1, \ldots, H\}\)

\[
\lambda^k \prod_{h=1}^{H} (x^h_i)^{\gamma_{kh}} = \sum_{j=1}^{N} K^k_{ij} \left[ \prod_{h=1}^{H} (x^h_j)^{\beta_{kh}} \right]
\]

\(x^h_i\) is the type \(h\) equilibrium variable (e.g. wage, price, labor...) in location/sector \(i\).

- Total number of variables to be solved \(H \times N, N \geq 2\)
- \(\lambda^k\) the ‘eigenvalue’ of the system. Its role across models varies.
- \(K^k_{ij} \geq 0\): exogenous linkages (e.g. trade/commuting costs, productivities...)
- \(\gamma_{kh}, \beta_{kh}\) are (exogenous) global parameters (e.g. EoS, parameters from distributions, spillovers...) and \(\Gamma, B\) the corresponding matrices
Theorem: Allen Arkolakis Li ’14

Theorem

Consider the system of equations (3).

If $\Gamma$ is invertible then:

(i) If $K_{ij}^k > 0$, then there exists a strictly positive solution, $\{x_i^h, \lambda^k\}$.

Define $A \equiv B\Gamma^{-1}$, with element $A_{ij}$ & $A^p \equiv \{|A_{ij}|\}$

(ii) If $K_{ij}^k \geq 0$ and the maximum of the eigenvalues of $A^p$ (spectral radius), $\rho(A^p) \leq 1$, then there exists at most one strictly positive solution (up-to-scale)
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$\rho(A^p) \leq 1$, then there exists at most one strictly positive solution
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(iii) If $\rho(A^p) < 1$ and $K_{ij}^k > 0$ for all $k, i, j$, the unique solution can be computed by a simple iterative procedure.
Theorem: Allen Arkolakis Li ’14

Consider the system of equations (3).

If $\Gamma$ is invertible then:

(i) If $K_{ij}^k > 0$, then there exists a strictly positive solution, $\{x_i^h, \lambda^k\}$.

Define $A \equiv B\Gamma^{-1}$, with element $A_{ij}$ & $A^p \equiv \{|A_{ij}|\}$

(ii) If $K_{ij}^k \geq 0$ and the maximum of the eigenvalues of $A^p$ (spectral radius), $\rho(A^p) \leq 1$, then there exists at most one strictly positive solution (up-to-scale)

(iii) If $\rho(A^p) < 1$ and $K_{ij}^k > 0$ for all $k, i, j$, the unique solution can be computed by a simple iterative procedure.

(iv) If $\rho(A^p) > 1$ and all elements of each column of $A$ have the same sign, then there exists a kernel $K_{ij}^k > 0$ such that there are multiple strictly positive solutions, i.e. for some set of frictions, the uniqueness conditions above are both necessary and sufficient.
Application in the Urban Model

- The theorem comes quite handy for the urban model with spatial spillovers
  - You can prove that existence is always guaranteed (not trivial)
  - With no spatial spillovers ($\eta = 0$) uniqueness holds with sufficiently large $\theta$
  - With no trade costs, uniqueness also holds as long if $|\eta(\sigma - 1)| \leq 1$

Note to grad students: can we revisit Davis Weinstein '02 '08 insights to test for multiple equilibria using gravity models?
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Welfare

- What about welfare?

- Standard formula (Arkolakis, Costinot, Rodriguez-Clare ’12) applies
  - Ex-post welfare \( d \ln W_j = -\frac{d \ln \lambda_{ij}}{\epsilon} \)
  - But in the case of labor mobility welfare equalizes; not so useful

- In addition, we want to consider comparative statics
Welfare Comparative Statics

- Can we further characterize welfare comparative statics?
  - In trade model easy to derive (Atkeson Burstein ’10 Burstein Cravino ’15):
    \[
    \frac{d \ln W}{d \ln \tau_{ij}} = -\lambda_{ij} \frac{E_j}{\sum_k E_k} = \frac{X_{ij}}{\sum_k E_k}
    \]

  where \( W \) is expenditure weighted welfare
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- In AAL we show that in economic geography ($W_j = W$) with no externalities, $\tilde{\alpha} = \tilde{\beta} = 0$, same result holds!
  - i.e. first order effect of $d \ln \tau_{ij}$ only depends on bilateral trade/world GDP!
  - In general, comparative static of welfare fully pinned down by data
  - Extremely useful results because it allows us to consider planning policies
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- In general, comparative static of welfare fully pinned down by data
- Extremely useful results because it allows us to consider planning policies
  - E.g. what is the optimal city structure?
Roadmap

- Analytical Model and Mapping to the Data
- Gravity, Modules, and Models
- Characterization of Urban Equilibrium
- Applications
Applications

- Basically, hundreds of applications undertaken with this setup in trade.
  - New wave of applications in economic geography, urban (AA, Ahlfedlt et al, Monte et al, AAL, Caliendo Parro Rossi-Hansberg ’14, Faber Gaubert ’15 etc)
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- Can we use this setup to think about trade cost/commuting costs etc?
  - Fast marching method (AA) ideally fit for the job
The Fast Marching Method for Spatial Economics
The Fast Marching Method for Spatial Economics

\{ j \mid t(i,j) = C \}
The Fast Marching Method for Spatial Economics
The Fast Marching Method with an Example
The Fast Marching Method with an Example
Transportation infrastructure
Conclusion

- We developed an analytical GE framework with tight connection to data
  - We showed that we can go a long way characterizing this setup
  - Robust and appealing framework: works as well for trade, geography, urban

- What are the future applications?
  - Can we think of the impact on inequality? (Burstein Vogel Morales ’15, Lee ’15, Gale et al ’15)
  - Non-CES? (Parenti et al ’15, Okubo Picard Thisse ’10, Arkolakis et al)
  - Spatial sorting? (Costinot Vogel ’10, Davis Dingel ’15, Ziv ’15, Gaubert)
  - How to allocate funds in building roads? What is the optimal city zoning?
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- Bottom line: Roback framework is extremely versatile (e.g Diamonds ’15)
  - But lacks spatial interactions. Is it high time for a new spatial framework?